Lecture Note: Cosmology II*

Masahiro Kawasaki

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Contents

0. Notation and units

In this lecture, Greek indices of vectors and tensors refer to four space-time coordinates, and Latin ones refer to three spatial coordinates.

$$\mu, \nu, \ldots = 0, 1, 2, 3 \tag{0.1}$$

$$i, j, \ldots = 1, 2, 3. \tag{0.2}$$

We use the time-like convention for the metric tensor, that is

$$g_{\mu\nu} = (+, -, -, -,) \tag{0.3}$$

For units, we use so-called natural units. In the natural units the light velocity c, the Planck constant $\hbar$ and the Boltzmann constant k_B are all set to unity,

$$c = \hbar = k_B = 1. \tag{0.4}$$

There is one fundamental dimension which is energy. Furthermore we take GeV as energy unit ($1\text{GeV} = 1.6022 \times 10^{-10}\text{J}$). Thus, the standard units of mass, length, time and temperature are related to GeV in the following way:

- Mass:

$$\text{g} = 5.61 \times 10^{23}\,\text{GeV}, \tag{0.5}$$

$$\text{GeV} = 1.78 \times 10^{-24}\,\text{g}. \tag{0.6}$$

- Length:

$$\text{cm} = 5.07 \times 10^{13}\,\text{GeV}^{-1}, \tag{0.7}$$

$$\text{GeV}^{-1} = 1.97 \times 10^{-14}\,\text{cm}. \tag{0.8}$$

- Time:

$$\text{sec} = 1.52 \times 10^{24}\,\text{GeV}^{-1}, \tag{0.9}$$

$$\text{GeV}^{-1} = 6.58 \times 10^{-25}\,\text{sec}. \tag{0.10}$$

- Temperature:

$$\mathrm{K} = 8.62 \times 10^{-14}\,\mathrm{GeV}, \tag{0.11}$$

$$\mathrm{GeV} = 1.16 \times 10^{13}\,\mathrm{K}. \tag{0.12}$$

For example, the mass of the sun $M_\odot$ is given by $M_\odot = 1.989 \times 10^{33}\,\mathrm{g} = 1.12 \times 10^{57}\,\mathrm{GeV}$. One important scale is the Planck mass which is defined as

$$M_G = 1/\sqrt{8\pi G} = 2.4 \times 10^{18}\,\mathrm{GeV}. \tag{0.13}$$

1. Standard Cosmology

1.1. Homogeneous and isotropic space

The standard cosmology based on the following two observational facts.

- The universe is spatially homogeneous on large scales. That means that there is no preferred locations in the universe.
- The universe is spatially isotropic on large scales. That means that there are no preferred directions in the universe.

Of coures on small scales we can see inhomogenities such as stars, galaxies and clusters of galxies. However, if we smooth the universe over about 100 Mpc, the universe looks homogeneous and isotropic. This was called cosmological principle in the past when the observations were limited. Now we can observed our universe on scales as large as $O(1000)$ Mpc.

There are three types of spatially homogeneous and isotropic space, 1) flat Euclidean space, 2) space with constant positive curvature and 3) space with constant negative curvature. It is obvious that the Euclidean space is homogeneous and isotropic.

1.1.1. Flat space

In a flat Euclidian space the element of length $d\ell$ between (x, y, z) and $(x+dx, y+dy, z+dz)$ is given by the familiar Pythagorean theorem as

$$d\ell^2 = dx^2 + dy^2 + dz^2. \tag{1.1}$$

In the polar coordinates (r, θ, ϕ), x, y and z are written as

$$z = ar\cos\theta, \tag{1.2}$$

$$x = ar\sin\theta\cos\phi, \tag{1.3}$$

$$y = ar\sin\theta\sin\phi, \tag{1.4}$$

where we introduce the scale parameter a for later convenience. Then the element of length (or spatial metric) is given by

$$d\ell^2 = a^2 \left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]. \tag{1.5}$$

1.1.2. Space with positive constant curvature

Space with constant positive curvature is constructed by considering a sphere in 4 dimensional Euclidean space. Using Cartesian coordinates (x, y, z, w), the 3 dimensional surface of the sphere with radius a is described by

$$x^2 + y^2 + z^2 + w^2 = a^2. \tag{1.6}$$

Since 4 dimensional space is Euclidean, the spatial metric $d\ell^2$ is given by

$$d\ell^2 = dx^2 + dy^2 + dz^2 + dw^2. \tag{1.7}$$

The surface of the sphere (1.6) is expressed by the following polar coordinates:

$$w = a\cos\chi, \tag{1.8}$$

$$z = a\sin\chi\cos\theta, \tag{1.9}$$

$$x = a\sin\chi\sin\theta\cos\phi, \tag{1.10}$$

$$y = a\sin\chi\sin\theta\sin\phi. \tag{1.11}$$

With use of the above polar coordinate Eq.(1.7) is written as

$$d\ell^2 = a^2 \left[d\chi^2 + \sin\chi^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]. \tag{1.12}$$

Defining r as $r = \sin\chi$, we obtain

$$dr = \cos\chi d\chi = \sqrt{1-\sin^2\chi}d\chi = \sqrt{1-r^2}d\chi, \tag{1.13}$$

which leads to

$$d\ell^2 = a^2 \left[\frac{dr^2}{1-r^2} + r^2\left(d\theta^2 + \sin^2\theta\ d\phi^2\right)\right]. \tag{1.14}$$

This is the spatial metric which describe homogeneous and isotropic space with positive curvature. The space is compact and hence called closed.

1.1.3. Space with negative constant curvature

The space with negative constant curvature is constructed by embedding a hyperbola

$$x^2 + y^2 + z^2 - w^2 = -a^2, \tag{1.15}$$

in 4 dimensional psude-Euclidean space with metric

$$d\ell^2 = dx^2 + dy^2 + dz^2 - dw^2. \tag{1.16}$$

This space is obtained from the constant positive curvature space (Eqs.(1.6) and (1.7)) by the following substitution:

$$a \to ia, \tag{1.17}$$

$$w \to iw, \tag{1.18}$$

$$\chi \to i\chi. \tag{1.19}$$

Thus, we obtain the spatial metric which describe homogeneous and isotropic space with negative curvature as

$$d\ell^2 = a^2 \left[\frac{dr^2}{1+r^2} + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right) \right]. \tag{1.20}$$

This space extends infinitely and the universe with such space is called open.

1.1.4. Robertson-Walker metric

Now we know homogeneous and isotropic space and taking into account that our universe is expanding, the space-time metric of the universe is given by

$$\boxed{ds^2 = dt^2 - a(t)^2 \left[\frac{dr^2}{1-Kr^2} + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right) \right]}. \tag{1.21}$$

This is called Robertson-Walker metric. Here $a(t)$ is the scale factor which represents the size of the universe. K represents the spatially curvature as

$$K = \begin{cases} 1 & \text{closed universe with positive curvature} \\ 0 & \text{flat universe} \\ -1 & \text{open universe with negative curvature} \end{cases}. \tag{1.22}$$

From Robertson-Walker metric (1.21) the Ricci tensor is calculated as

$$R_{00} = -3\frac{\ddot{a}}{a}, \tag{1.23}$$

$$R_{ij} = -\left[\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{K}{a^2} \right] g_{ij}, \tag{1.24}$$

where $\dot{} \equiv d/dt$. The Riccu scalar is given by

$$\mathcal{R} = -6\left[\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{K}{a^2}\right]. \tag{1.25}$$

Let us consider the spatial distance between two points A and B whose coordinates are A$(t, 0, 0, 0)$ and B$(t, r, 0, 0)$. (Without loss of generality, A can be taken as the origin from homogeneity of the space and we can take angular coordinates of B $= 0$ from spatially isotropy.) The physical distance between A and B is

$$d_p = \int_{\mathrm{A}}^{\mathrm{B}} d\ell = a(t)\int_0^r dr' \frac{1}{\sqrt{1 - Kr'^2}}. \tag{1.26}$$

This is called "proper distance". Notice that the proper distance is given by (coordinate distance)$\times$ (scale factor).

Here we derive the Hubble law which says that distant galaxies go away from us with velocities proportional to their distances. Suppose that the distance to some galaxy is d given by

$$d = a(t)\int_0^r \frac{dr'}{\sqrt{1 - Kr^2}}. \tag{1.27}$$

If the galaxy is comoving with the cosmic expansion, i.e. their spatial coodinates are constant, the recession velocity is

$$v = \dot{d} = \dot{a}\int_0^r \frac{dr'}{\sqrt{1 - Kr^2}} = \frac{\dot{a}}{a}d. \tag{1.28}$$

This shows the distance d is proportional to the recession velocity v with proportional factor $\dot{a}/a$ which is called Hubble parameter H. The Hubble parameter is defined as

$$\boxed{H(t) \equiv \frac{\dot{a}(t)}{a(t)}}. \tag{1.29}$$

Please notice that it is time-dependent. Sometimes the present Hubble parameter is called Hubble constant H_0 whose observed value is

$$H_0 = (67.4 \pm 1.4)\ \mathrm{km/s/Mpc}. \tag{1.30}$$

The present Hubble parameter is often expressed in units of 100 km/s/Mpc as

$$h \equiv H_0/(100\ \mathrm{km/s/Mpc}). \tag{1.31}$$

So the observed value of h is $h = 0.674 \pm 0.014$.

Another important consequence of the Robertson-Walker metric is that the wavelength of light increases as the universe expands. Suppose that light emitted at $r = r_e$ ($\theta = \phi = 0$) and $t = t_e$ observed at $r = 0$ and $t = t_0$ (the present time). From the light geodesics, $d^2s = 0$, and Eq. (1.21)

$$\frac{dt}{a(t)} = -\frac{dr}{\sqrt{1 - Kr^2}}. \tag{1.32}$$

Thus, we obtain

$$\int_{t_e}^{t_0} \frac{dt}{a(t)} = -\int_{r_e}^{0} \frac{dr'}{\sqrt{1 - Kr'^2}} = \int_{0}^{r_e} \frac{dr'}{\sqrt{1 - Kr'^2}} \tag{1.33}$$

Similarly, for light which is emitted at $t = t_e + \delta t_e$ and $r = r_e$ and observed at $r = 0$ and $t = t_0 + \delta t_0$,

$$\int_{t_e+\delta t_e}^{t_0+\delta t_0} \frac{dt}{a(t)} = \int_{0}^{r_e} \frac{dr'}{\sqrt{1 - Kr'^2}}. \tag{1.34}$$

Subtracting Eq. (1.33) from the above equation and assuming $\delta t_e \ll t_e$ and $\delta t_0 \ll t_0$, we find

$$\frac{\delta t_0}{a(t_0)} = \frac{\delta t_e}{a(t_e)}. \tag{1.35}$$

If we take $\delta t_e (\delta t_0)$ as a period [the time between successive wave crests] of the emitted (observed) light,

$$\frac{\delta t_e}{\delta t_0} = \frac{\nu_0}{\nu_e} = \frac{a(t_e)}{a(t_0)}, \tag{1.36}$$

where ν_e and ν_0 are the frequencies of the light at t_e and t_0. Thus, frequency ν of light decreases as the universe expands. In other words, wavelength λ of light increases as the universe expands ($\lambda \propto a$). The redshift z is defined as

$$\boxed{z \equiv \frac{\lambda_0 - \lambda}{\lambda} = \frac{a(t_0)}{a(t)} - 1}\,, \tag{1.37}$$

where λ and λ_0 the wavelengths of the light at t and t_0.

In quantum theory, the redshift means that the momentum p of a photon decreases as $1/a$ because the momentum is given by $p = 2\pi\hbar/\lambda$. Actually this applies to a generic particle with momentum p and mass m. Its momentum decreases as $p \propto 1/a$ in the expanding universe.

1.1.5. Energy momentum tensor

In the homogeneous and isotropic universe the energy momentum tensor takes the perfect fluid form:

$$T^{00} = \rho(t), \tag{1.38}$$

$$T^{ij} = -g^{ij}P(t), \tag{1.39}$$

or

$$T^{\mu}_{\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & -P & 0 & 0 \\ 0 & 0 & -P & 0 \\ 0 & 0 & 0 & -P \end{pmatrix}, \tag{1.40}$$

where ρ and P are the energy density and pressure of the universe. Using the velocity four vector $u^0 = 1$, $u^i = 0$ (four-velocity of an observer who is comoving with the cosmic expansion), the energy momentum tensor $T^{\mu\nu}$ is also written as

$$T^{\mu\nu} = -Pg^{\mu\nu} + (\rho + P)u^{\mu}u^{\nu}. \tag{1.41}$$

1.2. Einstein equation

We are ready for considering the Einstein equation which is written as

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = 8\pi G T_{\mu\nu} = \frac{1}{M_G^2}T_{\mu\nu}, \tag{1.42}$$

where $G_{\mu\nu}$ is the Einstein tensor, G is the Newton constant and M_G is the reduced Planck mass ($= 1/\sqrt{8\pi G} = 2.4 \times 10^{18}$ GeV).

Here is a remark on the cosmological term originally introduced by Einstein in 1917. Including the cosmological term the Einstein equation is written as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1.43}$$

where Λ is the cosmological constant. However, the cosmological term can be absorbed in the energy momentum tensor if we introduce $T^{\Lambda}_{\mu\nu}$ which is given by

$$T^{\Lambda}_{\mu\nu} = -P_{\Lambda}g_{\mu\nu} + (\rho_{\Lambda} + P_{\Lambda})u_{\mu}u_{\nu}, \tag{1.44}$$

$$P_{\Lambda} = -\rho_{\Lambda} = -\frac{\Lambda}{8\pi G}. \tag{1.45}$$

Rereading $T_{\mu\nu} + T^{\Lambda}_{\mu\nu}$ as $T_{\mu\nu}$, Eq (1.43) is reduced to Eq.(1.42).

Now let us derive the basic equations which describe the dynamical evolution of the universe. From the $(0,0)$ component of the Einstein equation we obtain

$$G_{00} = 3\left(\frac{\dot{a}^2}{a^2} + \frac{K}{a^2}\right) = 8\pi G T_{00} = 8\pi G \rho, \tag{1.46}$$

which leads to

$$\boxed{\frac{\dot{a}^2}{a^2} + \frac{K}{a^2} = \frac{8\pi G}{3}\rho} \; . \tag{1.47}$$

This equation is called the Friedmann equation. From (i,j) component,

$$G_{ij} = \left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{K}{a^2}\right) g_{ij} = 8\pi G T_{ij} = -8\pi G P g_{ij}, \tag{1.48}$$

which leads to

$$\boxed{\ddot{a} = -\frac{4\pi G}{3}(\rho + 3P)a} \; . \tag{1.49}$$

Another useful equation is obtained from Eqs. (1.47) and (1.48) as follows. First, multiplying Eq. (1.47) by a^2 and differentiating it with respect to t,

$$2\dot{a}\ddot{a} = \frac{8\pi G}{3}(\rho a^2)^{\cdot} = \frac{8\pi G}{3a}(\rho a^3)^{\cdot} - \frac{8\pi G}{3}a\dot{a}\rho \tag{1.50}$$

Using Eq. (1.48) in LHS of the above equation we obtain

$$\boxed{(a^3\rho)^{\cdot} = -P(a^3)^{\cdot}} \; . \tag{1.51}$$

1.3. Density of the universe

As is seen from the Friedmann equation, the cosmic expansion is determined by the density of the universe. Let us introduce an equation of state which describes the relation between density and pressure as

$$\boxed{P = w\rho} \; , \tag{1.52}$$

where w is the parameter that specifies the equation of state. Using Eq. (1.52), Eq. (1.51) is written as

$$(a^3\rho)^{\cdot} = -w\rho(a^3)^{\cdot},$$

$$\Rightarrow \quad \frac{\dot{\rho}}{\rho} = -(1+w)\frac{(a^3)^{\cdot}}{a^3} \; . \tag{1.53}$$

Thus, we obtain

$$\rho \propto a^{-3(1+w)} . \tag{1.54}$$

This describes how the density (energy) component with w evolves as the universe expands. In cosmology there are three kinds of important density components: matter, radiation and dark energy.

- **Matter**
  Matter consists of non-relativistic particles which (mostly) are not in thermal equilibrium. Since the velocity of matter particles is small, their pressure is almost zero ($P = 0$), which means $w = 0$. So the matter density $\rho_{\rm M}$ evolves as

  $$\rho_{\rm M} \propto a^{-3} . \tag{1.55}$$

  This is understood as follows. Since the energy of a non-relativistic particle is given by its rest mass, the energy density is inversely proportional to the volume $\sim a^3$.

- **Radiation**
  Radiation consists of relativistic particles which are thermal or non-thermal. The equation of state for relativistic particle is given by $P = \rho/3$ ($w = 1/3$), so the radiation energy $\rho_{\rm R}$ evolves as

  $$\rho_{\rm R} \propto a^{-4} . \tag{1.56}$$

  The energy of a relativistic particle is given by its momentum which decreases by redshift or adiabatic expansion [see Eq. (1.109)] as $\sim a^{-1}$. This explains the extra factor of a^{-1} in Eq. (1.56) compared with Eq. (1.55).

- **Dark energy**
  Dark energy is an energy component that drives accelerated expansion of the universe. From Eq. (1.49), the accelerated expansion, $\ddot{a} > 0$, requires $\rho + 3P < 0$, which leads to

  $$w < -\frac{1}{3} . \tag{1.57}$$

  In particular, dark energy with $w = -1$ is called cosmological constant because it is equivalent to the cosmological term $-\Lambda g_{\mu\nu}$ introduced by Einstein as mentioned in Sec 1.2. The energy momentum tensor of the dark energy ρ_Λ with $w = -1$ is written as

  $$T_{\mu\nu} = \rho_\Lambda g_{\mu\nu} = \frac{\Lambda}{8\pi G} g_{\mu\nu} . \tag{1.58}$$

In this lecture we only consider the cosmological constant as dark energy because the recent observations strongly suggest the present universe is dominated by some dark energy and its equation of state is given by $w \simeq -1$. Then, from Eq. (1.54)

$$\rho_\Lambda \propto a^0 = \text{constant} . \tag{1.59}$$

Let us define the density parameter Ω as

$$\boxed{\Omega \equiv \frac{8\pi G\rho}{3H^2} = \frac{\rho}{\rho_c}} , \tag{1.60}$$

where ρ_c is the critical density given by

$$\rho_c = \frac{3H^2}{8\pi G} , \tag{1.61}$$

and its present value is $\rho_{c,0} = 1.054h^2 \times 10^4$ eV cm^{-3}. Using the density parameter the Friedmann equation (1.47) is rewritten as

$$H^2 + \frac{K}{a^2} = \Omega H^2 \tag{1.62}$$

$$\Rightarrow \quad (\Omega - 1)H^2 = \frac{K}{a^2} . \tag{1.63}$$

Therefore the curvature of the universe is related to the cosmic density as

$$\Omega \begin{cases} > 1 \\ = 1 \\ < 1 \end{cases} \quad \Leftrightarrow \quad K \begin{cases} +1 \\ = 0 \\ -1 \end{cases} \tag{1.64}$$

The present abundances of the density components are shown in Table 1.1 and the fraction of each component is shown in Fig. 1.1. It is seen that radiation (photons and neutrinos) has large number density but gives a negligible contribution to the energy density of the present universe. Among known particles the baryons have the largest energy density which amounts to about 5% of the critical density. As has been known for a long time, dark matter has a significant density of the universe. Matter (baryons and dark matter) accounts for 23% of the present energy density. The present universe is dominated by dark energy, which amounts to about 67% of the critical density. Surprisingly, the total density of (radiation, matter and dark energy) is equal to the critical density within observational errors ($\sim 1\%$). Therefore, our universe is almost flat.

Since the dark energy, matter and radiation evolve as a^0, a^{-3} and a^{-4} respectively, matter or radiation dominates the universe in the early universe depending on the scale

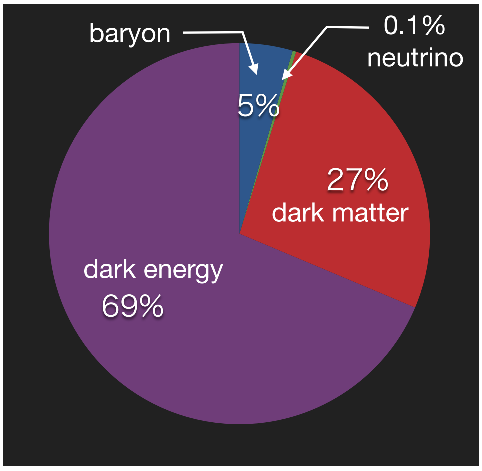

Figure 1.1.: Fractions of the density components.

Components	Temperature (K)	Number Density (cm^{-3})	energy density (eV cm^{-3})	Ω_0
photon (γ)	2.73	415	0.23	4.8×10^{-5}
neutrino (ν)	1.95	113×3	0.052×3	$1.09 \times 10^{-5} \times 3$
baryon (B)	—	2.5×10^{-7}	235	0.049
dark matter (DM)	—	—	1.16×10^3	0.265
dark energy (DE)	—	—	3.85×10^3	0.686

Table 1.1.: Present density components

factor as shown Fig. 1.2. Let us estimate the epoch t_* when densities of dark energy and matter are equal as

$$\Omega_{\text{M0}} \left(\frac{a_*}{a_0} \right)^{-3} = \Omega_{\Lambda 0}, \tag{1.65}$$

where $a_* = a(t_*)$, Ω_{M0} and $\Omega_{\Lambda 0}$ are density parameters of matter and dark energy (=cosmological constant). Since $\Omega_{M0} = 0.314$ and $\Omega_{\Lambda 0} = 0.686$, we obtain $a_* = 0.77 a_0$. In the same way, we can estimate the epoch t_{eq} at which matter density equals radiation one as

$$\Omega_{\text{M0}} \left(\frac{a_{\text{eq}}}{a_0} \right)^{-3} = \Omega_{\text{R0}} \left(\frac{a_{\text{eq}}}{a_0} \right)^{-4}, \tag{1.66}$$

where $a_{\text{eq}} = a(t_{\text{eq}})$ and Ω_{R0} is the density parameter of the present radiation and $\Omega_{\text{R0}} = 8.1 \times 10^{-5}$. Thus, we obtain $a_{\text{eq}} = 2.6 \times 10^{-4} a_0$.

In summary, there are three eras in the history of the standard universe.

- $a > a_*$: dark energy dominated universe (DED).
- $a_* > a > a_{\text{eq}}$: matter dominated universe (MD).

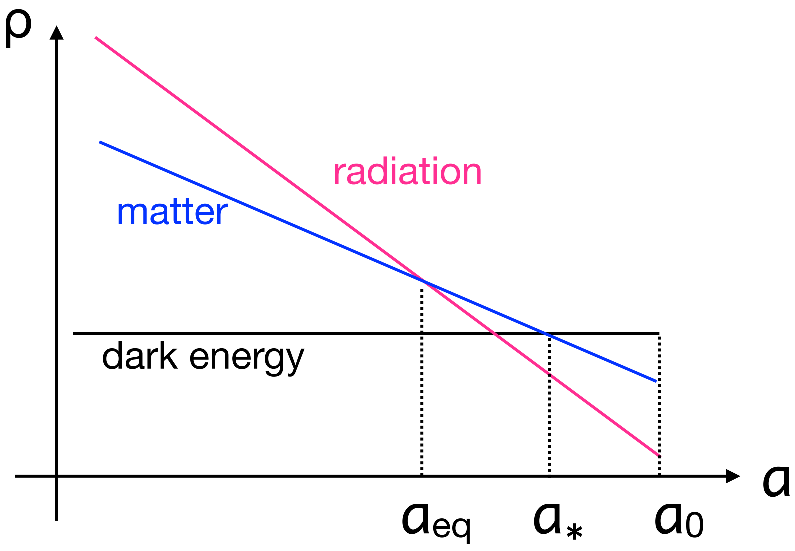

Figure 1.2.: Evolution of the density components.

- $a_{\rm eq} > a$: radiation dominated universe (RD).

1.3.1. (Remark) Neutrino mass

In Table 1.1 neutrinos are assumed to be massless. However, experiments reveal that neutrinos have small masses. If neutrinos have masses, flavor eigenstates (eigenstates for weak interaction) $|\nu_e\rangle, |\nu_\mu\rangle, |\nu_\tau\rangle$ are different from mass eigenstates $|\nu_1\rangle, |\nu_2\rangle, |\nu_3\rangle$. Thus, when neutrinos are produced through weak interaction they are in a mixed state of mass eigenstates. Since the mass eigenstates propagate with different wavelengths, they interfere and change the state of the produced neutrinos. Thus, neutrinos change their flavors periodically during their flight, which is called neutrino oscillation.

The neutrino oscillation was first discovered in 1998 by SuperKamiokande which observed atmospheric neutrinos. The atmospheric neutrinos are produced in the atmosphere of the earth through interaction between cosmic rays (mainly protons) and atoms (Fig. 1.3). In this interactions many pions are produced and they decay into mu neutrinos and muons which further decays into electron neutrinos, mu neutrinos and electrons. Superkamiokande found that the mu neutrino flux coming from below is smaller than expected, which means that mu neutrinos change into tau neutrinos during their flight through the earth. From the observation of the atmospheric neutrinos it was found that

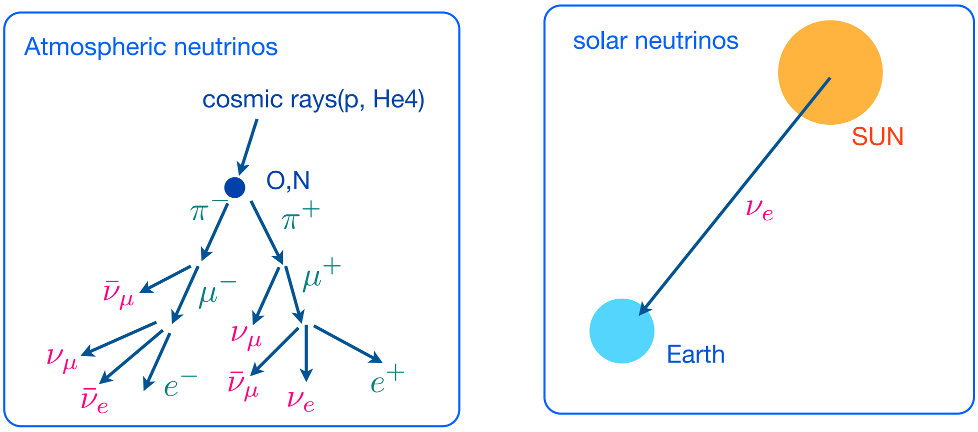

Figure 1.3.: Atmospheric and solar neutrinos

the mass squared difference between mu and tau neutrinos is

$$|m_3^2 - m_2^2| \simeq 2.5 \times 10^{-3} \text{ eV}^2. \tag{1.67}$$

The neutrino oscillation was also discovered for neutrinos emitted from the center of the sun (solar neutrinos). In this case electron neutrinos change into mu neutrinos and the mass squared difference between electron and mu neutrinos is

$$m_2^2 - m_1^2 \simeq 7.5 \times 10^{-5} \text{ eV}^2. \tag{1.68}$$

If neutrino masses are hierarchical like quarks, i.e. $m_3 \gg m_2 \gg m_1$, Eqs. (1.67) and (1.68) imply

$$m_3 \simeq 0.05 \text{ eV}, \qquad m_2 \simeq 8.7 \times 10^{-3} \text{ eV} , \tag{1.69}$$

from which the present density of neutrinos is estimated as

$$\rho_{\nu,0} \simeq 113 \text{ cm}^{-3} \times (0.05 + 0.0087) \text{ eV} \simeq 6.6 \text{ eV cm}^{-3} . \tag{1.70}$$

Since the present neutrino temperature $T_{\nu,0} \simeq 1.9\text{K} \simeq 1.6 \times 10^{-4}$ eV which is the typical momentum of the neutrinos is much smaller than the neutrinos masses, neutrinos are non-relativistic at present. However, their density parameter is

$$\Omega_{\nu,0} \simeq 1.4 \times 10^{-3} , \tag{1.71}$$

which is much smaller than $\Omega_{\text{DM},0}$ and $\Omega_{\text{B},0}$. On the other hand, neutrinos are relativistic for $T_\nu > m_\nu$ and their contribution to the total radiation density is significant. Therefore, in almost all cases we can consider neutrinos as radiation.

1.4. Solusions of Friedmann Equation

Let us solve Fiedmann equation (1.47). Here we write Friedmann equation as

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} = \frac{8}{3}\pi G\left(\rho_\Lambda + \rho_\mathrm{M} + \rho_\mathrm{R}\right). \tag{1.72}$$

Since our universe is found to be almost flat, we neglect the curvature term K/a^2.

Dark energy dominated universe

First, we consider the dark energy(=cosmological constant) dominated universe and for simplicity we neglect ρ_M and ρ_R. Then the above equation is given by

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8}{3}\pi G\rho_\Lambda. \tag{1.73}$$

Since $\rho_\Lambda =$ const., the equation is easily solved and we ontain

$$a = a_0 \exp\left[\sqrt{\frac{8\pi G}{3}\rho_\Lambda}\,(t - t_0)\right]. \tag{1.74}$$

So the universe expands exponentially when the cosmological constant dominates the universe.

Matter dominated universe

In matter dominated universe ($\rho_\mathrm{M} \gg \rho_\mathrm{R}, \rho_\Lambda$) the Friedmann equation is written as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8}{3}\pi G\rho_M = \frac{8}{3}\pi G\rho_{\mathrm{M},0}\left(\frac{a_0}{a}\right)^3 = H_0^2\Omega_{\mathrm{M},0}\left(\frac{a_0}{a}\right)^3, \tag{1.75}$$

from which we obtain the following solution:

$$a = a_0\left(\frac{3}{2}H_0\sqrt{\Omega_{\mathrm{M},0}}\right)^{2/3} t^{2/3}. \tag{1.76}$$

Here we set the boundary condition as $a(t\to 0) = 0$. Please notice that in MD era the scal factor $a(t)$ evolve as

$$\boxed{a(t) \propto t^{2/3}}\,. \tag{1.77}$$

Matter or cosmological constant dominated universe

The more accurate formula is obtained when both matter and cosmological constant give significan contributions to the total cosmic density as in the present universe. Neglecting only the radiation component, the Friedmann equation is written as

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\Omega_{\mathrm{M},0}\left(\frac{a(t)}{a_0}\right)^{-3} + H_0^2\Omega_{\Lambda,0} \tag{1.78}$$

$$\Rightarrow \quad \frac{\dot{a}}{a} = H_0\left[\Omega_{\Lambda,0}\left(\frac{a(t)}{a_0}\right)^3 + \Omega_{\mathrm{M},0}\right]^{1/2}\left(\frac{a(t)}{a_0}\right)^{-3/2}. \tag{1.79}$$

Defining $x \equiv a/a_0$,

$$\frac{dx}{dt} = H_0 x^{-1/2}[\Omega_{\Lambda,0}x^3 + \Omega_{\mathrm{M},0}]^{1/2}, \tag{1.80}$$

which is integrated with boundary condition $a(t \to 0) = 0$ as

$$\int_0^{a/a_0} \frac{x^{1/2}}{\sqrt{\Omega_{\Lambda,0}x^3 + \Omega_{\mathrm{M},0}}} = H_0 t. \tag{1.81}$$

Using the following formula

$$\int \frac{y^{1/2}dy}{\sqrt{y^3 + A}} = \frac{2}{3}\log(\sqrt{y^3} + \sqrt{y^3 + A}), \tag{1.82}$$

we obtain

$$\boxed{a(t) = a_0\left(\frac{\Omega_{\mathrm{M},0}}{\Omega_{\Lambda,0}}\right)^{1/3}\sinh^{2/3}\left[\frac{3}{2}\sqrt{\Omega_{\Lambda,0}}H_0 t\right]}\,. \tag{1.83}$$

In the limiting cases of $t \gg H_0^{-1}$ and $t \ll H_0^{-1}$, the above solution is estimated as

$$\frac{a}{a_0} \simeq \left(\frac{\Omega_{\mathrm{M},0}}{4\Omega_{\Lambda,0}}\right)^{1/3}\exp\left[\sqrt{\Omega_{\Lambda,0}}H_0 t\right] \qquad (t \gg H_0^{-1}) \tag{1.84}$$

$$\frac{a}{a_0} \simeq \left(\frac{3}{2}\sqrt{\Omega_{\mathrm{M},0}}H_0 t\right)^{2/3} \qquad (t \ll H_0^{-1}), \tag{1.85}$$

which correspond to the solutions (1.74) and (1.76), respectively.

The present age of the universe t_0 is estimated by setting $t = t_0$ in Eq. (1.83),

$$1 = \left(\frac{\Omega_{\mathrm{M},0}}{\Omega_{\Lambda,0}}\right)^{1/3}\sinh^{2/3}\left[\frac{3}{2}\sqrt{\Omega_{\Lambda,0}}H_0 t_0\right], \tag{1.86}$$

which leads to

$$t_0 = \frac{2}{3}\frac{H_0^{-1}}{\sqrt{\Omega_{\Lambda,0}}}\sinh^{-1}\left[\sqrt{\frac{\Omega_{\Lambda,0}}{\Omega_{\mathrm{M},0}}}\right]. \tag{1.87}$$

Using the observed values $H_0^{-1} = 1.45 \times 10^{10}$ yr, $\Omega_{\Lambda,0} = 0.69$ and $\Omega_{\mathrm{M},0} = 0.31$, the present age of the universe is estimated as $t_0 = 1.38 \times 10^{10}$ yr.

Radiation dominated universe

In radiation-dominated universe the Friedmannn equation is written as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_{\rm R}. \tag{1.88}$$

Since in most cases relativistic particles in radiation-dominated era are in thermal equilibrium, it is convenient to write $\rho_{\rm R}$ as a function of temperature. The energy density of a relativistic particle "i" in thermal equilibrium with temperature T_i is given by

$$\begin{aligned}\rho_i &= \frac{g_i}{(2\pi)^3}\int d^3p \frac{p}{e^{p/T_i}\pm 1} \qquad (+\text{ boson}, \quad -\text{ fermion}), \\ &= \begin{cases} \frac{\pi^2}{30} g_i T_i^4 & (\text{boson}) \\ \frac{7}{8}\frac{\pi^2}{30} g_i T_i^4 & (\text{fermion}) \end{cases}\end{aligned} \tag{1.89}$$

where g_i is the degrees of freedom (spin and particle-antiparticle). The extra factor 7/8 for fermionic particles is derived in the following way. Let us define $I_{\rm B}^{(n)}$ and $I_{\rm F}^{(n)}$ with n integer ($n \geq 2$) as

$$I_{\rm B}^{(n)} = \int_0^\infty \frac{x^n dx}{e^x - 1}, \qquad I_{\rm F}^{(n)} = \int_0^\infty \frac{x^n dx}{e^x + 1}. \tag{1.90}$$

Subtracting $I_{\rm F}^{(n)}$ from $I_{\rm B}^{(n)}$,

$$I_{\rm B}^{(n)} - I_{\rm F}^{(n)} = \int_0^\infty \frac{2x^n dx}{e^{2x}-1} = \frac{1}{2^n}\int_0^\infty \frac{y^n dy}{e^y - 1} = \frac{1}{2^n} I_{\rm B}^{(n)}, \tag{1.91}$$

where we have used $y = 2x$ in the second eqaulity. Fron the above equation we obtain the above relation betqween $I_{\rm B}^{(n)}$ and $I_{\rm F}^{(n)}$ as

$$\boxed{I_{\rm F}^{(n)} = \left(1 - \frac{1}{2^n}\right) I_{\rm B}^{(n)}}\, . \tag{1.92}$$

For $n = 3$ (case of energy density) the factor 7/8 is obtained.

The total radiation density is written as

$$\boxed{\rho_{\rm R} = \frac{\pi^2}{30} g_* T^4}\, , \tag{1.93}$$

where T is the photon temperature and g_* is the total relativistic degrees of freedom,

$$\boxed{g_* = \sum_{\rm boson} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8}\sum_{\rm fermion} g_i \left(\frac{T_i}{T}\right)^4}\, , \tag{1.94}$$

Temperatute	g_*
$< m_e$	$2 + \frac{21}{4}\left(\frac{4}{11}\right)^{4/3}$
$m_e - m_\mu$	$\frac{43}{4}$
$m_\mu - m_\pi$	$\frac{57}{4}$
$m_\pi - T_{\rm H}$	$\frac{69}{4}$
$T_{\rm H} - m_c$	$\frac{247}{4}$
$m_c - m_\tau$	$\frac{289}{4}$
$m_\tau - m_b$	$\frac{303}{4}$
$m_b - m_{W,Z}$	$\frac{345}{4}$
$m_{W,Z} - m_h$	$\frac{381}{4}$
$m_h - m_t$	$\frac{385}{4}$
$m_t - T_{\rm EW}$	$\frac{427}{4}$
$> T_{\rm EW}$	$\frac{427}{4}$

Table 1.2.: Relativistic degrees of freedom. $T_{\rm H}$ and $T_{\rm WS}$ are the temperatures of qurak-hadron and electro-weak phase transitions.

Most particles in the radiation have the same temperature as photons, i.e. $T_i = T$, but some particles that are decoupled from the thermal bath at early epochs have different temperatures. For example, at temperature $T = 1$ MeV the relativistic particles whose masses are lighter than T are photons(γ), electrons (positrons) ($e^\pm$) and three species of neutrinos (3ν). They contribute to g_* as

$$g_* = \overbrace{\underset{\text{(helicity)}}{2}}^{\gamma} + \overbrace{\underset{\text{(fermion)}}{\frac{7}{8}} \times \underset{\text{(spin)}}{2} \times \underset{(e^\pm)}{2}}^{e} + \overbrace{\underset{\text{(fermion)}}{\frac{7}{8}} \times \underset{(3\nu)}{3} \times \underset{(\nu\bar{\nu})}{2}}^{\nu} = \frac{43}{3}. \tag{1.95}$$

For the particle content of the standard model of paricle physics, g_* is caluculated as shwon in Table 1.2.

In considering the cosmological evolution of the radiation density and temperature, it is useful to introduce the entropy S. In thermodynamics we have

$$dS = \frac{1}{T}d(\rho V) + \frac{P}{T}dV, \tag{1.96}$$

where V is the volume of the system we consider. We assume that the energy density and pressure are functions of T only. Then, from Eq. (1.96)

$$\frac{\partial S}{\partial V} = \frac{1}{T}(\rho + P) \tag{1.97}$$

$$\frac{\partial S}{\partial T} = \frac{V}{T}\frac{d\rho}{dT}, \tag{1.98}$$

from which we obtain the integrability condition as

$$\frac{\partial^2 S}{\partial T \partial V} = \frac{\partial}{\partial T}\left(\frac{1}{T}(\rho + P)\right) = \frac{\partial}{\partial V}\left(\frac{V}{T}\frac{d\rho}{dT}\right) . \tag{1.99}$$

Thus, we obtain

$$dP = \frac{1}{T}(\rho + P)dT , \tag{1.100}$$

which is applied to Eq. (1.96) as

$$\begin{aligned} dS &= \frac{1}{T}d\left[(\rho + P)V\right] - \frac{V}{T}dP = \frac{1}{T}d\left[(\rho + P)V\right] - \frac{V}{T^2}(\rho + P)dT \\ &= d\left[\frac{(\rho + P)V}{T}\right] . \end{aligned} \tag{1.101}$$

So Eq. (1.96) can be integrated as

$$S = \frac{V}{T}\left(\rho(T) + P(T)\right). \tag{1.102}$$

In the case of the universe, the volume is set to equal to a^3 and the entropy of the universe is given by

$$dS = \frac{1}{T}d(\rho a^3) + \frac{P}{T}da^3, \tag{1.103}$$

$$S = \frac{a^3}{T}\left(\rho(T) + P(T)\right). \tag{1.104}$$

Furthermore, we define the entropy density s as $s = S/a^3$. Since $P = \rho/3$, s is given by

$$\boxed{s = \frac{4\rho}{3T} = \frac{2\pi^2}{45}g_{s*}T^3} , \tag{1.105}$$

where g_{s*} is the relativistic degrees of freedom for entropy,

$$\boxed{g_{s*} = \sum_{\text{boson}} g_i\left(\frac{T_i}{T}\right)^3 + \frac{7}{8}\sum_{\text{fermion}} g_i\left(\frac{T_i}{T}\right)^3} . \tag{1.106}$$

From the Einstein equation which describe the evolution of the energy density [Eq. (1.51)] given by

$$\frac{d(\rho a^3)}{dt} = -P\frac{d(a^3)}{dt}, \tag{1.107}$$

and Eq. (1.103), we obtain

$$\frac{dS}{dt} = 0. \tag{1.108}$$

Therefore the entropy is conserved and hence the universe expands adiabatically, which leads to

$$g_{s*}T^3a^3 = \text{consant}. \tag{1.109}$$

When g_{s*} is regard as constant, the cosmic temperature T is proportional to $1/a$.

We are now ready to solve the Friedmann equation (1.88). Since $T \propto 1/a$ and $\rho_{\rm R}$ is given by Eq. (1.93) the equation is written as

$$-\frac{\dot{T}}{T} = \left(\frac{\pi^2 g_*}{90}\right)^{1/2}\frac{T^2}{M_{\rm G}}, \tag{1.110}$$

where we have used $M_{\rm G} \equiv 1/\sqrt{8\pi G} \simeq 2.4\times 10^{18}$ GeV. The above equation is easily solved and we obtain

$$t = \left(\frac{2\pi^2 g_*}{45}\right)^{-1/2}\frac{M_{\rm G}}{T^2}, \tag{1.111}$$

$$= 2.3 \text{ sec } g_*^{-1/2}\left(\frac{T}{10^{10}\text{K}}\right)^{-2}, \tag{1.112}$$

$$= 1.7 \text{ sec } g_*^{-1/2}\left(\frac{T}{\text{MeV}}\right)^{-2}. \tag{1.113}$$

Notice that in radiation-dominated universe the scale factor a evolves as

$$\boxed{a(t) \propto 1/T \propto t^{1/2}}\,. \tag{1.114}$$

1.5. Thermal History of the Universe

1.5.1. Summary

In Fig. 1.4 summary of the history of the universe predicted in the standard cosmology is shown. In the standard cosmology the evolution of the universe after about 1 sec is well understood and the following several important events take place from 1 sec to the present:

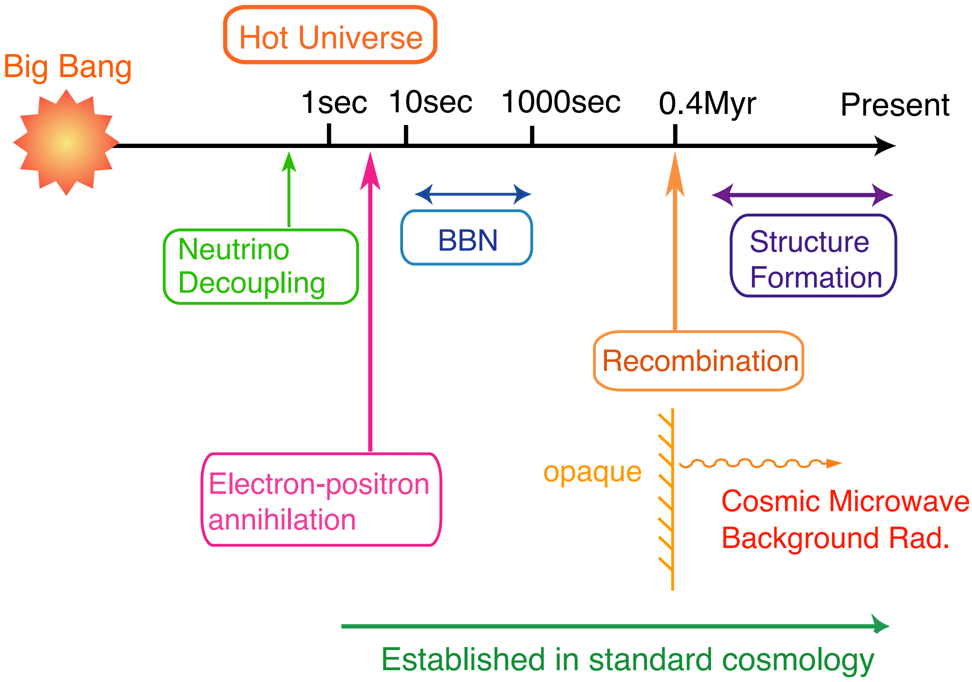

Figure 1.4.: Thermal hisory of the universe.

- *Neutrino decoupling*
  In the early universe ($t \lesssim 0.3$ sec), neutrinos are in thermal equilibrium via weak interactions. However, when the temperature becomes as low as a few MeV ($t \sim 0.3$ sec) the interaction rate is not large enough to keep neutrinos in thermal equilibrium and neutrinos are decoupled from the thermal bath.

- *Electron-positron annihilation*
  When the temperature T is larger than the electron mass m_e, electrons and positrons are as abundant as photons. They are annihilated and pair-created and their densities are given by the thermal values. However, at $T \lesssim m_e$ electrons and positrons cannot be pair-created and only annihilations proceed. As a result positrons disappear in the universe and a small fraction of electrons remain owing to charge neutrality.

- *Big Bang Nucleosynthesis (BBN)*

From $t \sim 1$ sec to $t \sim 10^3$ sec light elements like D, ^{3}He and ^{4}He are synthesized.

- Recombination
  At $t \simeq 0.4$ Myr protons and electrons form bound systems, i.e hydrogens, which is called recombination although this is the first time for them to combine. After that, photons can freely streams without scattering with electrons and are presently observed as the cosmic microwave radiations.

- *Structure formartion*
  After recombination the large scaled structure of the universe such as galaxies and clusters is formed from tiny density fluctuations through gravitational instability.

1.5.2. Neutrino decoupling

In the early universe ($T \gtrsim 2$ MeV) neutrinos are in thermal equilibrium via the following weak interaction:

$$\nu_i + \bar{\nu}_i \longleftrightarrow e^+ + e^- \qquad (i = e, \mu, \tau) \tag{1.115}$$

The number density of the neutrino n_i evolves according to the Boltzmann equation,

$$\frac{dn_{\nu_i}}{dt} + 3Hn_{\nu_i} = -\langle \sigma v \rangle \left(n_{\nu_i}^2 - n_{\nu \mathrm{eq}}^2 \right), \tag{1.116}$$

where $\langle \sigma v \rangle$ is the thermal averaged cross section and $n_{\nu_i \mathrm{eq}}^2$ is the equilibrium number density. The neutrino number density is determined by competition between reaction rate Γ and the cosmis expansion rate H. The rate of the weak interaction is given by

$$\Gamma = \langle \sigma v \rangle \simeq \frac{4G_F^2 \langle E^2 \rangle}{9\pi} \frac{3\zeta(3)}{2\pi^2} T^3 \sim G_F^2 T^5 \tag{1.117}$$

where G_F is the Fermi coupling constant $\simeq 1.17 \times 10^{-5}$ GeV^{-2}, E the energy of neutrinos and $\zeta(3)(= 1.202\ldots)$ is the zeta function of 3. On the other hand the expansion rate

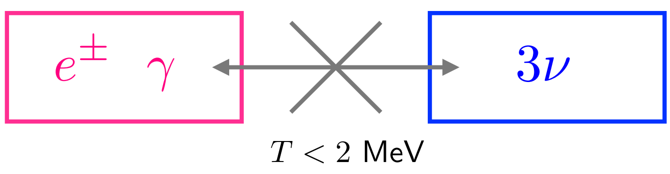

Figure 1.5.: Neutrino decoupling.

(=Hubble parameter) is given by

$$H = \frac{T^2}{M_G}\left(\frac{\pi^2 g_*}{90}\right)^{1/2} \tag{1.118}$$

Therefore at high temperatute $\Gamma \gg H$ and hence neutrinos are in thermal equilibrium $n_{\nu_i} = n_{\nu\mathrm{eq}}$ while nutrinos are diluted by the cosmic expansion $n_\nu \propto a^{-3}$ without interacting at low temperature. Thus, when $\Gamma \simeq H$ neutrinos decouple from the thermal bath. The decoupling temperature T_d determined from $\Gamma \sim G_F^2 T_d^5 \sim T_d^2/M_G \sim H$ is

$$\boxed{T_d \simeq 2\ \mathrm{MeV}}\ . \tag{1.119}$$

So the neutrino sector and the photon-electron sector decouple and no energy exchange happens (Fig. 1.5).

1.5.3. Neutrinos after decoupling

Let us consider their momentum distribution $f_\nu(p)$. At temperature higher than or equal to the decoupling temperature ($T \geq T_d$) $f_\nu(p)$ obeys the Fermi-Dirac distribution given by

$$f_\nu(p) = \frac{1}{\exp\left(\frac{p}{T}\right) + 1}\ . \tag{1.120}$$

However, at $T < T_d$ neutrinos decouple from the thermal bath and they freely stream in the universe. Thus, the momentum of each neutrino is redshifted and hence decreases as $p \propto 1/a$. Thus, the momentum at $t > t_d$ (t_d: decoupling time) is given by

$$p = p_d \frac{a(t_d)}{a(t)}\ , \tag{1.121}$$

where $p_d \equiv p(t_d)$. Since the distribution at $t = t_d$ is given by

$$f_\nu(p_d) = \frac{1}{\exp\left(\frac{p_d}{T_d}\right) + 1}\ . \tag{1.122}$$

we obtain the distribution function at $t > t_d$ as

$$f(p) = \frac{1}{\exp\left(\frac{p}{T_d}\frac{a(t)}{a(t_d)}\right) + 1}\ . \tag{1.123}$$

Here if we define the "neutrino temperature" T_ν as

$$T_\nu \equiv T_d \frac{a(t_d)}{a(t)}\ , \tag{1.124}$$

the momentum distribution is written as

$$f_\nu(p) = \frac{1}{\exp\left(\frac{p}{T_\nu}\right) + 1} \ . \tag{1.125}$$

Therefore, the neutrino distribution is the same as the thermal one with temperature T_ν. Please notice that the neutrino temperature T_ν is always proportinal to $1/a$ after the neutrino decoupling.

1.5.4. Electron positron annihilation

After the neutrino decoupling, when the cosmic temperature decreases as low as the electron mass $m_e (\simeq 0.511$ MeV), elecrons and positrons which are as abundant as photons start to annihilate each other,

$$e^+ + e^- \quad \longrightarrow \quad 2\gamma \ . \tag{1.126}$$

As a result almost all electrons and positrons disappear in the universe. From charge neutrality the number density of electrons is slightly lager than that of positrons and its difference is the same as the proton number density, namely,

$$n_{e^-} - n_{e^+} = n_p \ll n_\gamma \ . \tag{1.127}$$

Thus, a small number of electrons survive the annihilation.

Let us consider the effect of the $e^\pm$ annihilation on the photon and neutrino temperatures using entropy conservation. The entropy S_γ in the photon sector at $t = t_1$ before the annihilation is given by

$$S_\gamma = a_1^3 \, \frac{2\pi^2}{45} \left(2 + \frac{7}{8} \times 2 \times 2\right) T_1^3 = (a_1 T_1)^3 \, \frac{2\pi^2}{45} \, \frac{11}{2} \qquad (T_1 > m_e), \tag{1.128}$$

where $T_1 = T(t_1)$ and $a_1 = a(t_1)$. After $e^\pm$ annihilation the entropy is written as

$$S_\gamma = a_2^3 \, \frac{2\pi^2}{45} \, (2) \, T_2^3 = (a_2 T_2)^3 \, \frac{2\pi^2}{45} \, 2 \qquad (T_2 < m_e), \tag{1.129}$$

where t_2 is some time after the annihilation. Since the photon and neutrino sectors are decoupled, the entropy in each sector is conserved separately. The conservation of S_γ leads to

$$T_2 = T_1 \left(\frac{a_1}{a_2}\right) \left(\frac{11}{4}\right)^{1/3} \ . \tag{1.130}$$

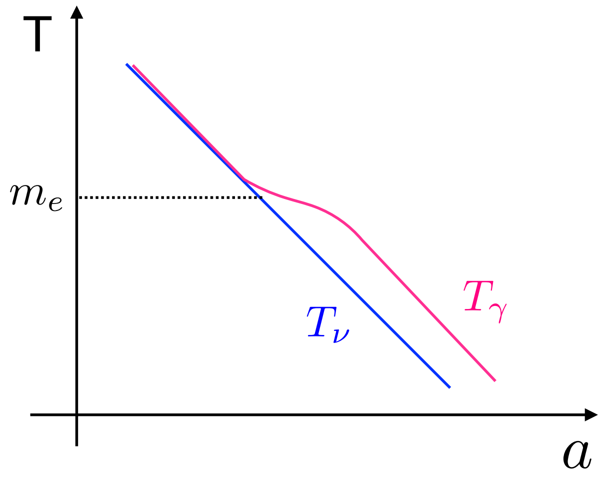

Figure 1.6.: Evolution of the photon and neutrino temperatures.

On the other hand, the neutrino sector is not affected by the $e^{\pm}$ annihilation, the conservation of its entropy S_ν gives the following relation between $T_\nu(t_1) = T_{\nu 1} = T_1$ and $T_\nu(t_2) = T_{\nu 2}$:

$$T_{\nu 2} = T_{\nu 1}\left(\frac{a_1}{a_2}\right) = T_1\left(\frac{a_1}{a_2}\right) . \tag{1.131}$$

From Eqs. (1.130) and (1.131) we obtain

$$\boxed{T_\nu = T\left(\frac{4}{11}\right)^{1/3}} . \tag{1.132}$$

Here notice that we always denote T as the photon temperature. The photon temperature relatively increases due to heating by $e^{\pm}$ annihilation while the neutrino temperature decreases as $T_\nu \propto 1/a$ (Fig. 1.6).

We know that the present photon temperature is $T = 2.726$ K from which the neutrino temperature is estimated as $T_\nu = 1.95$ K. Using the photon temperature the photon number density is calculated as

$$n_{\gamma,0} = \frac{2}{(2\pi)^3}\int_0^\infty 4\pi p^2 dp \frac{1}{\exp(p/T_0) - 1} = \frac{2\zeta(3)}{\pi^2}T_0^3. \tag{1.133}$$

Similary the neutrino number density per species is estimated as

$$n_{\nu,0} = \frac{3}{4}\frac{2\zeta(3)}{\pi^2}T_{\nu,0}^3 . \tag{1.134}$$

Using $T_0 = 2.73$ K and $T_{\nu,0} = 1.95$ K we obtain

$$n_{\gamma,0} = 415 \text{ cm}^{-3} \;, \tag{1.135}$$

$$n_{\nu,0} = 113 \text{ cm}^{-3} \;. \tag{1.136}$$

1.5.5. Big bang nucleosynthesis

Big bang nucleosynthesis is the process by which helium 4 (^{4}He) nuclei are synthesized from neutrons and protons in the early universe ($T \simeq 1$ MeV $-$ 10 keV),

$$2p + 2n \quad \longrightarrow \quad {}^4\text{He} \;. \tag{1.137}$$

In the process small amounts of other light elements like deuterium (D), helium 3 (^{3}He) and lithium 7 (^{7}Li) are also produced.

Fixing n/p ratio

At high temperature $T \gtrsim 1$ MeV, protons(neutrons) are changed to neutrons(protons) via the following weak interactions:

$$\nu_e + n \;\longleftrightarrow\; p + e^- \tag{1.138}$$

$$e^+ + n \;\longleftrightarrow\; p + \bar{\nu}_e \tag{1.139}$$

$$n \;\longleftrightarrow\; p + e^- + \bar{\nu}_e \tag{1.140}$$

The rate of the above reactions is given by $\Gamma \sim G_F^2 T^5$. When the reaction rate Γ is larger than the expansion rate H, the chemical equilibrium is established and we have

$$\mu_{\nu_e} + \mu_n = \mu_p + \mu_{e^-} \;, \tag{1.141}$$

where μ_i is the chemical potential of particle "i". The equilibrium number density of proton(neutron) is written as

$$n_{p(n)} = 2\left(\frac{m_{p(n)}T}{2\pi}\right)^{3/2} \exp\left[-(m_{p(n)} - \mu_{p(n)})/T\right] \;. \tag{1.142}$$

Thus, the neutron-to-proton ratio is

$$\left(\frac{n_n}{n_p}\right)_{\text{eq}} = \exp\left[-\frac{m_n - m_p - \mu_n + \mu_p}{T}\right] \;. \tag{1.143}$$

The electron chemical potential is related to the difference between number densities of electrons and positrons as

$$n_{e^-}-n_{e^+} = \frac{g_e}{2\pi^2}\int_0^\infty p^2 dp \left[\frac{1}{\exp((\sqrt{p^2+m_e^2}-\mu_e)/T)+1} - \frac{1}{\exp((\sqrt{p^2+m_e^2}+\mu_e)/T)+1}\right] , \tag{1.144}$$

where $\mu_e = \mu_{e^-} = -\mu_{e^+}$ and g_e is the spin degrees of freedom. When $T \gg m_e$

$$n_{e^-} - n_{e^+} = \frac{g_e}{6\pi^2}T^3\left[\pi^2\frac{\mu}{T} + \left(\frac{\mu}{T}\right)^2\right] . \tag{1.145}$$

From Eq. (1.127)

$$n_{e^-} - n_{e^+} \simeq \frac{\mu_e}{T}\frac{T^3}{3} \ll n_\gamma \sim T^3 , \tag{1.146}$$

which leads to $\mu_e/T \ll 1$. So the electron chemical potential is negligible in Eq. (1.141). Moreover, if we assume $\mu_{\nu_e}/T \ll T$, the condition for chemical equilibrium is written as

$$\mu_p = \mu_n , \tag{1.147}$$

from which Eq. (1.143) is rewritten as

$$\left(\frac{n_n}{n_p}\right)_{\rm eq} = \exp\left[-\frac{m_n - m_p}{T}\right] . \tag{1.148}$$

Since the neutron mass is slightly larger than the proton mass $m_n - m_p = 1.293$ MeV, the chemical equilibrium predicts that the neutron-to-proton ratio is less than 1. However, the weak interaction rate decreases as the universe cools down, and the weak interaction freezes out at $\Gamma \simeq H$. The freeze-out temperature is estimated as $T_f \simeq 0.7$ MeV, which leads to

$$\boxed{\left(\frac{n_n}{n_p}\right)_f = \exp\left[-\frac{m_n - m_p}{T_f}\right] \simeq \frac{1}{7}} . \tag{1.149}$$

Since most of neutrons existing at T_f form ^{4}He, the ^{4}He abundance Y_p is estimated as

$$Y_p \equiv \frac{\rho_{^4\text{He}}}{\rho_B} = \frac{2n_n}{n_n + n_p} \simeq 0.25 . \tag{1.150}$$

Thus, the 25% of baryons are synthesized to ^{4}He. Fig 1.7 show the precise calculation of the neutron-to-proton ratio.

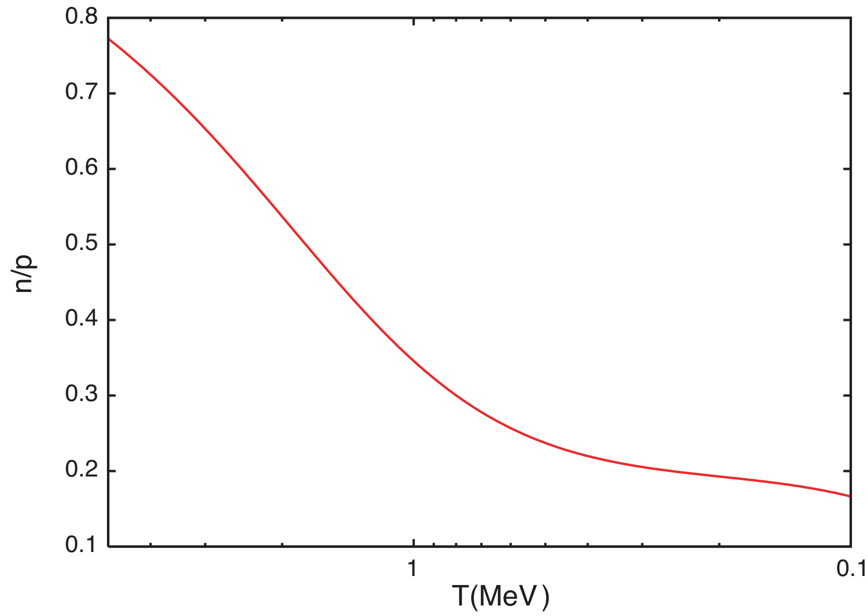

Figure 1.7.: Evolution of neutron-to-proton ratio.

Deuterium bottleneck

After the neutron-to-proton ratio is fixed by freeze-out of the weak interaction, the next step towards helium synthesis is formation of deuterons via

$$p + n \quad \longleftrightarrow \quad \mathrm{D} + \gamma \ . \tag{1.151}$$

At high temperature $T \gtrsim 0.1$ MeV, the background photons have energy high enough to destroy the synthesized deuterons whose binding energy Q_D is small ($Q_\mathrm{D} = 2.22$ MeV). Thus, the D production does not take place effectively and hence the nucleosynthesis does not proceed by this obstacle (deuterium bottleneck). The deuteron production proceeds when the temperature decreases as low as 0.1 MeV,

$$p + n \quad \longrightarrow \quad \mathrm{D} + \gamma \ . \tag{1.152}$$

Helium synthesis

After formation of D, ^{4}He is produced by rapid nuclear reactions, e.g. the following successive reactions:

$$\mathrm{D}+\mathrm{D} \longrightarrow {}^3\mathrm{He}+n \tag{1.153}$$
$${}^3\mathrm{He}+n \longrightarrow {}^3\mathrm{H}+p \tag{1.154}$$
$${}^3\mathrm{H}+\mathrm{D} \longrightarrow {}^4\mathrm{He}+n\ . \tag{1.155}$$

Through the reactions (1.152)–(1.155) most of neutrons existing at $T \sim 1$ MeV form ^{4}He nuclei and small numbers of D, ^{3}He and ^{3}H nuclei are produced as by-product. (Later, ^{3}H decays into $^3\mathrm{He}+e^-+\bar{\nu}_e$ with lifetime 17.8 year.)

Heavier elements are hardly produced in BBN because no stable nuclei with mass number $A = 5$ or 8 exist in nature and Coulomb barrier becomes significant. However, only a tiny amount of ^{7}Li nuclei are produced through the reaction,

$${}^4\mathrm{He}+{}^3\mathrm{H} \longrightarrow {}^7\mathrm{Li}+\gamma \tag{1.156}$$
$${}^4\mathrm{He}+{}^3\mathrm{He} \longrightarrow {}^7\mathrm{Be}+\gamma \tag{1.157}$$
$$\searrow {}^7\mathrm{Be}+e^- \longrightarrow {}^7\mathrm{Li}+\nu_e \tag{1.158}$$

The abundances of light elements depend only on baryon-to-photon ratio,

$$\boxed{\eta_B \equiv \frac{n_B}{n_\gamma}}\ . \tag{1.159}$$

Fig. 1.8 shows the theoretical prediction for abundances of ^{4}He, ^{3}He, D and ^{7}Li together with ranges of observed abundances. From this figure it is found that the BBN predictions for ^{4}He and D are consistent with the observed abundances for $\eta_B \simeq 6 \times 10^{-10}$. Therefore, the BBN can determine the baryon density of the universe. As for ^{7}Li abundance the BBN predicts too large a value if we take $\eta_B \simeq 6 \times 10^{-10}$, which is called "lithium problem".

1.5.6. Recombination

When the cosmic temperature is about 3000 K ($t \sim 0.38$ Myr), electron and protons combine to form hydrogen atoms as

$$p+e^- \longrightarrow \mathrm{H}+\gamma\ . \tag{1.160}$$

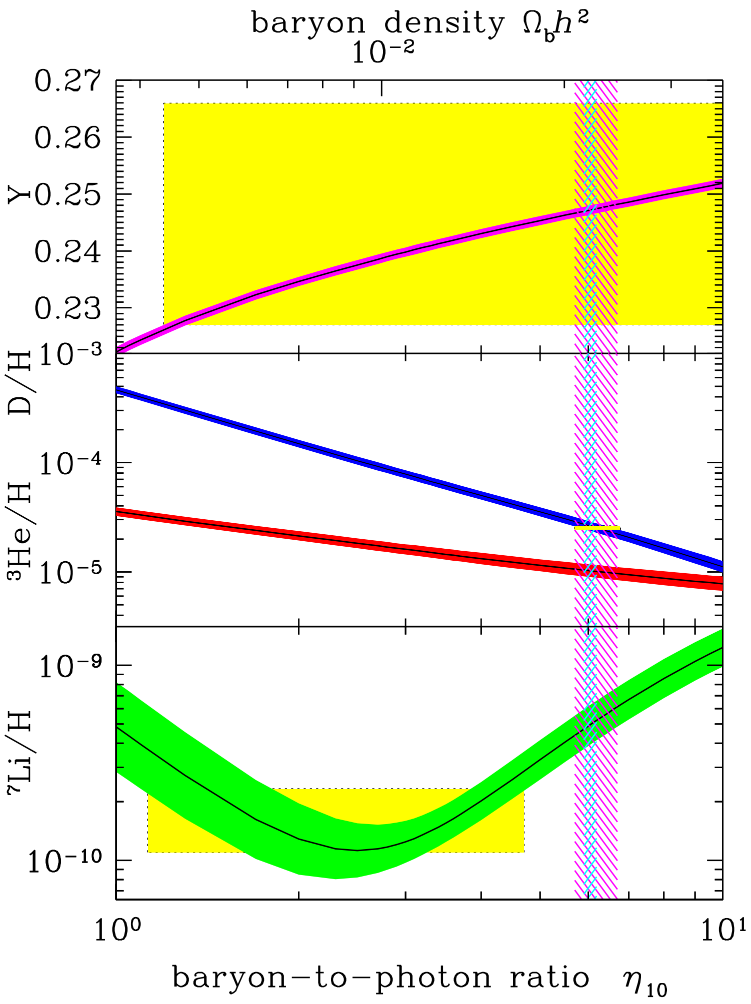

Figure 1.8.: Abundances of ^{4}He, ^{3}He, D and ^7Li predicted by BBN. The yellow bands shows 95% CL range of observed abundances [1].

This process is called recombination following the astrophysics convention although this is the first time for them to combine. Let us neglect ^{4}He for simplicity. Then,

$$n_B = n_p + n_{\rm H} \ , \tag{1.161}$$

$$n_p = n_e \ . \tag{1.162}$$

Since the chemical equilibrium is established at the beginning of the recombination, we have the relation among the chemical potentials as

$$\mu_{\rm H} = \mu_e + \mu_p \ . \tag{1.163}$$

Here notice that that the chemical potential for photons is zero. Since the electrons, protons and hydrogen atoms are non-relativistic, their number densities are given by

$$n_i = g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} \exp\left(\frac{\mu_i - m_i}{T}\right) \qquad i = e,\ p,\ \mathrm{H}\ , \tag{1.164}$$

where g_i is the spin degree of freedom ($g_p = g_e = 2, g_{\mathrm{H}} = 4$). From Eqs. (1.163) and (1.164) we obtain

$$\left(\frac{n_{\mathrm{H}}}{n_p n_e}\right)_{\mathrm{eq}} = \frac{g_{\mathrm{H}}}{g_e g_p}\left(\frac{m_e T}{2\pi}\right)^{-3/2} \exp\left(\frac{m_p + m_e - m_{\mathrm{H}}}{T}\right) = \left(\frac{m_e T}{2\pi}\right)^{-3/2} e^{B/T}\ , \tag{1.165}$$

where $B(= m_p + m_e - m_{\mathrm{H}} = 13.6$ eV) is the hydrogen binding energy.

Using the ionization fraction defined by

$$X \equiv \frac{n_p}{n_B} = \frac{n_e}{n_B}\ , \tag{1.166}$$

Eq. (1.165) is rewritten as

$$\frac{1 - X_{\mathrm{eq}}}{(X_{\mathrm{eq}})^2} = n_B \left(\frac{m_e T}{2\pi}\right)^{-3/2} e^{B/T}\ . \tag{1.167}$$

Since the baryon number density n_B is written as

$$n_B = \eta_B n_\gamma = \frac{2\zeta(3)}{\pi^2} T^3 \eta_B\ . \tag{1.168}$$

the ionization fraction satisfies the well-known Saha formura,

$$\boxed{\frac{1 - X_{\mathrm{eq}}}{(X_{\mathrm{eq}})^2} = \frac{4\sqrt{2}\zeta(3)}{\sqrt{\pi}} \eta_B \left(\frac{T}{m_e}\right)^{3/2} e^{B/T}}\ . \tag{1.169}$$

Figure 1.9 shows the evolution of the electron fraction $X_e = n_e/(n_H + n_p) \simeq X$. It is seen that the recombination takes place around $T \sim 4000$ K$= 0.4$ eV ($z \sim 1300$). The temperature T_{rec} when the recombination takes place is significantly lower than the hydrogen binding energy 13.6 eV. This comes from the fact that the number density of photons is much larger than that of baryons. Even if hydrogen atoms are formed the background photons with energy larger than B can ionize them, and such high energy photons are sufficiently abundant unless the temperature is much lower than the binding energy. Thus, recombination does not proceed effectively until the temperature becomes as low as ~ 0.4 eV.

Figure 1.9.: Evolution of the ionization fraction. The fraction X_e is larger than 1 at high temperature because ^{4}He is taken into account.

As recombination proceeds the free electrons and protons rapidly decreases and hence the recombination rate becomes smaller than the cosmic expansion rate. In that case the Saha formula is no longer satisfied and the ionization fraction is frozen out. Since the recombination and the expansion rates are proportional to $Xn_B \propto X\Omega_B h^2$ and $\rho^{1/2} \propto (\Omega_{\rm M0}h^2)^{1/2}$, respectively. So the freeze-out ionization fraction X_f is proportional to $\Omega_{\rm M0}^{1/2}/(\Omega_B h)$, more precisely

$$X_f \simeq 3 \times 10^{-5}\,\frac{\Omega_{\rm M0}^{1/2}}{\Omega_B h}\ . \tag{1.170}$$

Because free electrons almost disappear due to recombination, the mean free time $\tau_{\rm T}$ of the background photons for Thomson scattering becomes long as

$$\tau_{\rm T} = \frac{1}{\sigma_{\rm T} n_e} \simeq 4 \times 10^{12}\ {\rm sec}\ \frac{1}{X}\left(\frac{T}{10^3{\rm K}}\right)^{-3}\ , \tag{1.171}$$

where $\sigma_{\rm T}(= 6.6 \times 10^{-25}{\rm cm}^2)$ is Thomson cross section. The mean free time is longer than the cosmic time which is given by

$$t \simeq \frac{2}{3}H_0^{-1}\Omega_{\rm M0}^{-1/2}\left(\frac{T}{T_0}\right)^{-3/2} \simeq 8 \times 10^{13}\ {\rm sec}\left(\frac{T}{10^3{\rm K}}\right)^{-3/2}\ , \tag{1.172}$$

for $T \lesssim 10^3$ K and $X \sim 10^{-4}$. Here we have used Eq. (1.76) and $T \propto 1/a$. As a result the background photons freely streams without scattering off the residual background electrons and they are observed as the cosmic microwave background (CMB).

2. Inflationary Universe

2.1. Problems of the Standard Big Bang Model

As we have seen the previous chapter, the standard big bang model is very successful in describing our universe at $t \gtrsim 1$ sec. However, if the standard big bang model is applied to the very early universe it is confronted with several problems listed below.

- Flatness problem
- Large entropy problem
- Horizon problem
- Monopole problem
- Gravitino problem
- Origin of the density fluctuations

Most of theses problems are found to be solved by the inflationary universe.

2.1.1. Flatness problem

The flatness problem comes from the observational fact that the present universe is close to flat about 13.8 billion years after the big bang. This is quite unnatural if you consider how the flatness of our universe evolves in time.

Let us begin with the Friedmann equation,

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} = \frac{8\pi G}{3}\rho \ . \tag{2.1}$$

Using the Hubble parameter $H = \dot{a}/a$ and the density parameter $\Omega = (8/3)\pi G\rho H^{-2}$, the above equation is written as

$$H^2 + \frac{K}{a^2} = \Omega H^2 \ . \tag{2.2}$$

Since $\Omega = 1$ corresponds to the flat universe, we can define the "flatness" parameter as

$$\Omega - 1 = \frac{K}{a^2 H^2} \ . \tag{2.3}$$

From Eqs. (1.77) and (1.114) a^2H^2 evolves as

$$a^2 H^2 = \dot{a}^2 \propto \begin{cases} a^{-1} \propto T & \text{(MD)} \\ a^{-2} \propto T^2 & \text{(RD)} \end{cases} \ , \tag{2.4}$$

in matter or radiation dominated universe, respectively. Thus, the flatness $\Omega - 1$ evolves as

$$\Omega - 1 \propto \begin{cases} a \propto T^{-1} & \text{(MD)} \\ a^2 \propto T^{-2} & \text{(RD)} \end{cases} \ . \tag{2.5}$$

Since we know that at present $\Omega_0 \lesssim 0.01$, we can obtain the flatness at the Planck time ($= M_G^{-1}$) which is the earliest time when the classical description (i.e., Einstein equation) can be applied to the universe,

$$|\Omega - 1| \lesssim 0.01 \underbrace{\left(\frac{T_0}{T_{\text{eq}}}\right)}_{\text{MD}} \underbrace{\left(\frac{T_{\text{eq}}}{T_{\text{pl}}}\right)^2}_{\text{RD}} = 0.01 \ \left(\frac{10^{-13}\text{GeV}}{10^{-9}\text{GeV}}\right) \left(\frac{10^{-9}\text{GeV}}{10^{18}\text{GeV}}\right)^2 \sim 10^{-60} \ , \tag{2.6}$$

where T_{eq} is the temperature at the matter-radiation-equality time. Eq. (2.6) shows that the universe should be extremely flat with accuracy 10^{-60} at the Planck time. This requires an unnatural fine tuning.

2.1.2. Large entropy problem

Let us estimate the entropy $\tilde{S}$ inside a sphere with curvature radius of the universe.[1] Since the curvature radius is given by $a/\sqrt{|K|}$, $\tilde{S}$ is written as

$$\tilde{S} \simeq \left(\frac{a}{\sqrt{|K|}}\right)^3 s = \left[\frac{1}{H^2|\Omega - 1|}\right]^{3/2} s = \left[\frac{1}{H_0^2|\Omega_0 - 1|}\right]^{3/2} s_0 \ , \tag{2.7}$$

where we have used Eq.(2.3) and entropy conservation ($\tilde{S} = \tilde{S}_0$).

The present entropy density is estimated as

$$s_0 = \frac{2\pi^2}{45}\left(2\ T_{\gamma,0}^3 + \frac{7}{8} \times 2 \times 3\ T_{\nu,0}^3\right) \tag{2.8}$$

$$= 1.715 T_{\gamma,0}^3 = 2.8 \times 10^3 \ \text{cm}^{-3} \tag{2.9}$$

[1] In our convention $\tilde{S} = S$ since $|K| = 1$. However, we can rescale a, r and K as $a \to \beta a$, $r \to \beta^{-1} r$ and $K \to \beta^2 K$ (β: a constant) without changing Robertson-Walker metric. In that case K takes an arbitrary value and the spatial curvature radius is given by $a/\sqrt{|K|}$.

With use of the present Hubble radius $H_0^{-1} \simeq 4000$ Mpc $= 1.3 \times 10^{28}$ cm and $|\Omega_0 - 1| \lesssim 0.01$, we obtain

$$\tilde{S} \gtrsim 10^{91} . \tag{2.10}$$

Therefore, our universe has unnaturally huge amount of entropy. This is the large entropy problem. It is noticed that the large entropy problem has the same origin as the flatness problem because both are based on Eq.(2.3).

2.1.3. Horizon problem

Horizon

There are two types of horizons in cosmology; one is the particle horizon and the other is the event horizon. The particle horizon $\ell_{\rm H}$ is the maximum travel distance of light from $t = 0$ to t. The geodesics of light is given by $ds^2 = 0$. From the Robertson-Walker metric [Eq. (1.21)]

$$ds^2 = 0 = dt^2 - a(t)^2 \frac{dr^2}{1 - Kr^2} \quad \Rightarrow \quad \frac{dr}{\sqrt{1 - Kr^2}} = \frac{dt}{a(t)} , \tag{2.11}$$

where we assume that the light travels in the $\phi = \theta = 0$ direction. Thus, the particle horizon is given by

$$\boxed{\ell_{\rm H}(t) = a(t) \int_0^{r_{\rm H}} \frac{dr}{\sqrt{1 - Kr^2}} = a(t) \int_0^t \frac{dt'}{a(t')}} , \tag{2.12}$$

where $r_{\rm H}$ is the coordinate distance for the particle horizon. For $a(t) \propto t^m$ $(0 < m < 1)$, we obtain

$$\ell_{\rm H} = \frac{t}{1 - m} = \begin{cases} 2t & ({\rm RD}) \\ 3t & ({\rm MD}) \end{cases} . \tag{2.13}$$

As is seen from the definition the particle horizon is the maximum distance within which causal relations are established and hence it is very important in cosmology.

The event horizon $\ell_{\rm He}$ is the maximum travel distance of light from t to $t_{\rm max}$. $t_{\rm max}$ is the maximum time if the universe lasts for a finite time or infinity if the universe exists forever. $\ell_{\rm He}$ is given by

$$\boxed{\ell_{\rm He}(t) = a(t) \int_t^{t_{\rm max}} \frac{dt'}{a(t')}} . \tag{2.14}$$

A finite $\ell_{\rm He}$ is obtained for de Sitter universe whose scale factor evolve as a $\exp(Ht)$ with H constant,

$$\ell_{\rm He} = e^{Ht} \int_t^{\infty} dt' e^{-Ht'} = \frac{1}{H} . \tag{2.15}$$

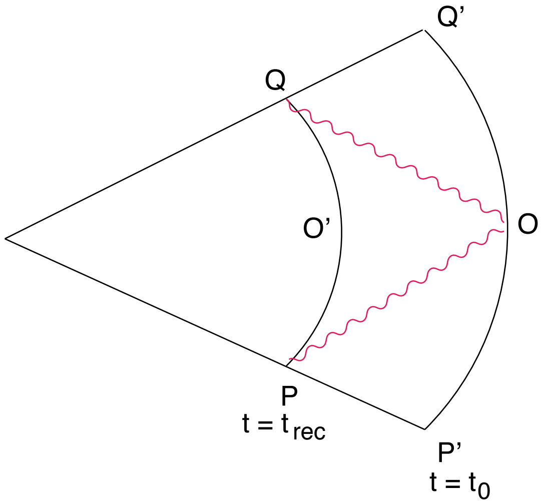

Figure 2.1.: Horizon problem.

If some event happens outside of the event horizon at the time $t' > t$, we never know the event.

Another useful scale related to the horizon is the Hubble radius given by H^{-1} which is written as

$$H^{-1}(t) = \frac{t}{m} = \begin{cases} 2t & \text{(RD)} \\ 3t/2 & \text{(MD)} \end{cases} , \tag{2.16}$$

for the universe with $a \propto t^m$. So the Hubble radius is roughly equal to the particle horizon if $a \propto t^m$, and equal to the event horizon for the de Sitter universe.

Horizon problem

The horizon problem is closely related the fact that the observed CMB radiation is highly isotropic, which apparently violates causality. Suppose CMB photons coming from the opposite directions. These photons are emitted from the space-time points P and Q at the recombination epoch $t_{\rm rec}$ and observed by the observer at O as shown in Fig. 2.1. In the figure O' is the point at $t_{\rm sec}$ with same spatial coordinate as O. Using Eq. (2.11), the proper distance PO' is given by

$$d_{\rm PO'} = a(t_{\rm rec}) \int_{t_{\rm rec}}^{t_0} \frac{dt'}{a(t')} \simeq t_{\rm rec}^{2/3} \int_{t_{\rm rec}}^{t_0} \frac{dt'}{t'^{2/3}} \simeq 3\; t_{\rm rec}^{2/3}\; t_0^{1/3} , \tag{2.17}$$

where we assume that the universe is matter dominated from $t_{\rm rec}$ to t_0, and used $t_0 \gg t_{\rm rec}$ at the last equality. From symmetry the distance between P and Q is $d_{\rm PQ} = 6 t_{\rm rec}^{2/3} t_0^{1/3}$.

On the other hand, the particle horizon at $t_{\rm rec}$ is

$$d_{\rm H} = 3\ t_{\rm rec}\ . \tag{2.18}$$

The ratio $d_{\rm PQ}/d_{\rm H}$ is evaluated as

$$\frac{d_{\rm PQ}}{d_{\rm H}} = 2\left(\frac{t_0}{t_{\rm rec}}\right)^{1/3} \simeq 2\left(\frac{T_{\rm rec}}{T_0}\right)^{1/2} \simeq 2\left(\frac{3000{\rm K}}{2.7{\rm K}}\right)^{1/2} \simeq 74\ . \tag{2.19}$$

Therefore, the points P and Q which are far away and have no causal relation emit photons with same intensity. This is unnatural and called the horizon problem.

2.1.4. Monopole problem

So far we have considered the rather conceptual problems in the standard big bang model. The next problem we discuss is the monopole problem which is a more practical one. The standard model of particle physics is based on gauge theory with $SU(3)\times SU(2)\times U(1)$ symmetry, where $SU(2)\times U(1)$ is the symmetry of the electroweak theory unifying the weak and electromagnetic interactions. The idea of unification naturally leads to the grand unified theories (GUTs) which unify the strong and electroweak interactions within the framework of a gauge field theory based on a symmetry group G e,g, $SU(5)$ or $SO(10)$. It is expected that the group G is spontaneously broken to $SU(3)\times SU(2)\times U(1)$ at low energy by Higgs mechanism. In general when a spontaneous symmetry breaking takes place topological defects are produced through the Kibble mechanism. The topological defects are classified to domain walls, strings and monopoles which are two, one and zero dimensional objects, respectively. When $G \to SU(3)\times SU(2)\times U(1)$ occurs monopoles are formed.

To understand the spontaneous breaking let us consider a simple real scalar fields ϕ with potential

$$V(\phi) = \lambda(\phi^2 - v^2)^2 + cT^2\phi^2\ , \tag{2.20}$$

where the second term represents the finite temperature correction with c and λ constants. This potential has a Z_2 symmetry ($\phi \to -\phi$). At high temperature ($T \gg v$) the potential has the minimum at $\phi = 0$ (Fig. 2.2(a)), and Z_2 symmetry is not broken. On the other hand at low temperature ϕ takes v or $-v$ (Fig. 2.2(a)). The vacuum $\langle\phi\rangle = v$ is not invariant under $\phi \to -\phi$, so Z_2 is broken.

When the symmetry breaking occurs, some regions take the field value v and other regions takes $-v$ as shown in Fig, 2.2(b) because ϕ takes v or $-v$ with equal probability.

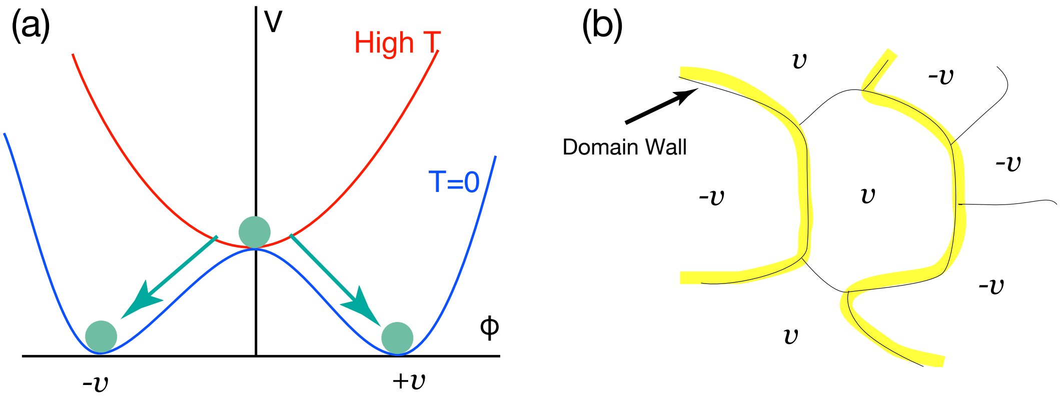

Figure 2.2.: (a) Potential $V(H)$ and (b) formation of domain walls.

Since the field should be continuous there should be a boundary region where the field value takes $\phi \sim 0$ and hence has a large potential energy. In the case of the real field with Z_2 the boundary region is two dimensional and called domain wall.

For a spontaneous symmetry breaking like $G \to SU(3) \times SU(2) \times U(1)$ a point-like defect called monopole is formed (Fig. 2.3). The size of the region where the scalar field aligns is called coherent length ξ. The monopole number density is larger for a shorter coherent length, which leads to

$$n_{\rm M} \sim \frac{1}{\xi^3} \; , \tag{2.21}$$

Let us estimate the cosmic monopole density. Since the coherent length cannot exceed the horizon, i,e, $\xi < \ell_{\rm H} = 2t_{\rm f}$, the number density at the formation epoch ($t_{\rm f}$) is given by

$$n_{\rm M} \gtrsim \frac{1}{8t_{\rm f}^3} \; . \tag{2.22}$$

After formation the monopoles are diluted as $\propto a^{-3}$. Since the entropy density $s(= 2\pi^2/45 \; g_* T^3$ also decreases as a^{-3}, the ratio $n_{\rm M}/s$ (s: entropy density) is constant and is given by

$$\frac{n_{\rm M}}{s} \gtrsim \frac{1}{8t_{\rm f}} \left(\frac{2\pi^2}{45} g_* T_{\rm f}^3 \right)^{-1} = \frac{\pi}{4\sqrt{90}} g_*^{1/2} \frac{T_{\rm f}^3}{M_G^3} \simeq 0.8 \left(\frac{T_{\rm f}}{M_G} \right)^3 \; , \tag{2.23}$$

Here we have used $t = (45/2\pi^2 g_*)^{1/2} M_G/T$. The phase transition takes place at the GUTs scale ($T_{\rm f} \simeq 10^{16}$ GeV), which leads to

$$\frac{n_{\rm M}}{s} \gtrsim 6 \times 10^{-8} \; . \tag{2.24}$$

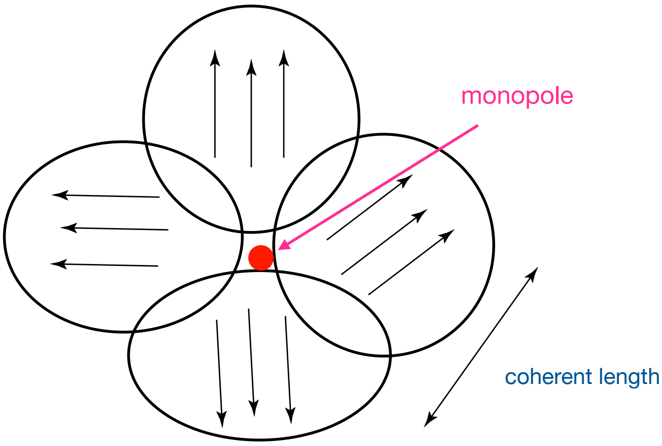

Figure 2.3.: Formation of a momopole.

At present $s_0 = 2.8 \times 10^3$ cm^{-3} and the monopole mass is $m_{\rm M} \simeq 10^{17}$ GeV (see, Sec. A.1), so the monopole density is given by

$$\rho_{\rm M} \gtrsim 1.6 \times 10^{13} {\rm GeV cm}^{-3} , \tag{2.25}$$

which is much larger than the present critical density $\rho_{c,0} \simeq 5 \times 10^{-6}$ GeV cm^{-3} and hence contradicts the obserbation. This is called the monopole problem.

2.1.5. Gravitino problem

One of promising ideas beyond the standard model of particle physics is supersymmetry which is a symmetry between bosons and fermions. In supersymmetry every bosonic (ferionic) particle in the standard model has its fermionic (bosonic) superpartner. For example,

$$\gamma \text{ (photon)} \iff \tilde{\gamma} \text{ (photino)} \tag{2.26}$$

$$g \text{ (gluon)} \iff \tilde{g} \text{ (gluino)} \tag{2.27}$$

$$e \text{ (electron)} \iff \tilde{e} \text{ (selectron)} \tag{2.28}$$

$$q \text{ (quark)} \iff \tilde{q} \text{ (squark)} \tag{2.29}$$

The gravitino is the sperpartner of the graviton which mediates gravity,

$$g_{\mu\nu} \text{ (graviton)} \iff \tilde{\psi}_\mu \text{ (gravitino)} . \tag{2.30}$$

In the standard big bang universe gravitinos are in thermal equilibrium at the Planck time and their number density $n_{2/3}$ is given by

$$n_{2/3} \sim n_\gamma \ . \tag{2.31}$$

The gravitino mass $m_{3/2}$ is expected to be $O(1)$ TeV in some class of models and the lifetime of the gravitino is very long because the gravitino interacts with other particles olny through gravity. The lifetime for $\psi_\mu \to \gamma + \tilde{\gamma}$ is

$$\tau_{3/2} \simeq 4 \times 10^5 \text{ sec} \left(\frac{m_{3/2}}{1\text{TeV}}\right)^{-3} , \tag{2.32}$$

Thus when the gravitino decays the ratio of the gravitino density to that of the background photons is estimated as

$$\left.\frac{\rho_{3/2}}{\rho_\gamma}\right|_{\text{decay}} \sim \left.\frac{m_{3/2} n_{3/2}}{T n_\gamma}\right|_{\text{decay}} \sim \frac{m_{3/2}}{T_{\text{decay}}} \sim \frac{1\text{TeV}}{\text{keV}} \gg 1 \ . \tag{2.33}$$

This means that huge entropy is produced by the gravitino decay, which dilutes the baryon density. Since the baryon density should be $n_B/n_\gamma \sim 10^{-10}$ at the BBN epoch, the present baryon density becomes much smaller than the observed value. This is the gravitino problem which was first pointed out by Steven Weinberg.

2.1.6. Origin of the density fluctuations

The large scale structures such as glaxies and clusters obserbed at present are thought to be formed from initial small density fluctuations which grows through gravitational instabilities. How are those density fluctuations created in the eraly universe? In oreder to get feeling about the epoch when the density fluctuations are produced let us consider the fluctuations with galaxy scale. The typical mass of a galaxy M_{gal} is about $10^{13} M_\odot$,

$$M_{\text{gal}} \sim 10^{13} M_\odot \ . \tag{2.34}$$

On the other hand, the matter mass inside the horizon M_{H} is given by

$$M_{\text{H}} = \rho_{\text{M}} \frac{4\pi}{3}(2t)^3 = \rho_{c,0}\Omega_{M,0}\left(\frac{T}{T_0}\right)^3 \frac{32\pi}{3} t^3 \ . \tag{2.35}$$

Here we assume that the universe is radiation-dominated. Using Eq. (1.111) we ontain

$$M_{\text{H}} = \frac{32\pi}{3}\left(\frac{45}{2\pi^2 g_*}\right)^{3/4} \frac{M_G^{3/2}\rho_{c,0}}{T_0^3}\Omega_{\text{M},0}\ t^{3/2} \tag{2.36}$$

$$\simeq 0.2 M_\odot \Omega_{\text{M},0}\left(\frac{t}{\text{sec}}\right)^{3/2} , \tag{2.37}$$

where $g_* = 2.34$. The galaxy scale becomes equal to the horizon scale when $M_{\rm gal} \sim M_{\rm H}$, which happens at $t_{\rm gal} \sim 4 \times 10^9$ sec. Since we do not know any mechanism to produce density fluctuations at $t \gtrsim t_{\rm gal}$, we should suppose that the density fluctuations are produced at $t \ll t_{\rm gal}$. This means that the size (or wavelength) of initial fluctuations responsible for galaxies is much larger than the horizon. However, it is impossible for some physical process to produce density fluctuations whose size is over-horizon. Therefore, it is difficult to explain the origin of the density fluctuations in the standard big bang model.

2.2. Success of inflationary universe

The most of the problems discussed are solved if there exist a period of accelerated expansion (= inflation) in the very early universe. The universe that experiences the period of accelerated expansion at its early stage is called inflationary universe.

Suppose that the universe is dominated by the vacuum energy $\rho_{\rm v}$. Here the vacuum energy is a term in quantum field theory which is the same as the dark energy with $w = -1$. In this case, taking into account that $\rho_{\rm v}$ is constant, the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho_{\rm v} \tag{2.38}$$

has a simple solution

$$\boxed{a(t) \propto \exp(H_{\rm inf} t), \qquad H_{\rm inf} = \left[\frac{8\pi}{3} G\rho_{\rm v}\right]^{1/2} = \frac{\rho_{\rm v}^{1/2}}{\sqrt{3} M_G}}\ . \tag{2.39}$$

Thus, the universe expands exponentially. There are the following two points in the inflationary universe:

- Existence of the quasi-exponential expansion (= inflation).
- Reheating of the universe after inflation.

Here reheating is the process where the vacuum energy converts to the hot radiation after inflation.

2.2.1. Flatness problem

Let us see how the inflationary universe solves the flatness problem. We suppose that the universe exponentially expands from t_i to t_f and it is radiation dominated after

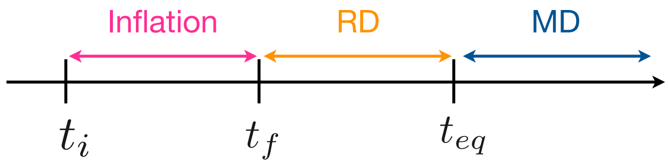

Figure 2.4.: Timeline of inflationary universe.

inflation until the equal-time $t_{\rm eq}$ after which the matter dominated universe follows (see Fig. 2.4). During inflation the scale factor expands as

$$a(t) = a_i \exp[H_{\rm inf}(t - t_i)] \ , \tag{2.40}$$

where $a_i = a(t_i)$. At the end of inflation the scale factor increases by

$$Z \equiv \frac{a_f}{a_i} = \exp[H_{\rm inf}(t_f - t_i)] = \exp[(H_{\rm inf}\Delta t)] \ . \tag{2.41}$$

Now let us calculate the flatness parameter $|\Omega - 1|$. If the flatness at t_i is $|\Omega - 1|_i$ the flatness at the end of inflation is given by

$$|\Omega - 1|_f = \left(\frac{a_i H_i}{a_f H_f}\right)^2 |\Omega - 1|_i \ , \tag{2.42}$$

where $H_i = H(t_i)$ and $H_f = H(t_f)$. Since the Hubble parameter H is constant during inflation, $H_i = H_f = H_{\rm inf}$, and we obtain

$$|\Omega - 1| = \left(\frac{a_i}{a_f}\right)^2 |\Omega - 1|_i = \frac{1}{Z^2}|\Omega - 1|_i \ . \tag{2.43}$$

After inflation the evolution of the flatness is the same as the standard universe, so the present flatness is estimated as

$$|\Omega - 1|_0 \simeq \left(\frac{T_f}{T_{\rm eq}}\right)^2 \left(\frac{T_{\rm eq}}{T_0}\right) |\Omega - 1|_f \tag{2.44}$$

$$= \left(\frac{10^{16}\text{GeV}}{10^{-9}\text{GeV}}\right)^2 \left(\frac{10^{-9}\text{GeV}}{10^{-13}\text{GeV}}\right) |\Omega - 1|_f = \frac{10^{54}}{Z^2}|\Omega - 1|_i \ , \tag{2.45}$$

where we tale $T_f = 10^{16}$ GeV. We obtain the present flatness $|\Omega - 1|_0 \sim O(0.01)$ for $H_{\rm inf}\Delta t \gtrsim 65$ even if $|\Omega - 1|_i \sim O(1)$. Hereafter, we call $H_{\rm inf}\Delta t$ the total e-folds of inflation denoted as $N_{\rm tot}$. More precisely, the total e-folds is given by

$$N_{\rm tot} = \int_{t_i}^{t_f} H\, dt = \int_{t_i}^{t_f} \frac{\dot{a}}{a} dt = \int_{t_i}^{t_f} \frac{d\ln a}{dt} dt = \ln\frac{a_f}{a_i} \ . \tag{2.46}$$

Thus, the flatness problem is solved in the inflationary universe. The point is that $\mathcal{H} \equiv H(t)a(t)$ is a increasing function during inflation.

2.2.2. Horizon problem

Next let us consider the horizon problem. For this end we consider the evolution of the region whose size $L_{\rm H}(t)$ is equal to the particle horizon at the beginning of the inflation t_i as

$$L_{\rm H}(t_i) \sim t_i \ . \tag{2.47}$$

This region is enlarged during inflation and its size at the end of inflation is given by

$$L_{\rm H}(t_f) \sim Z t_i \ . \tag{2.48}$$

After inflation the region is further enlarged by the cosmic expansion and at present

$$L_{\rm H}(t_0) \sim Z t_i \left(\frac{a(t_0)}{a(t_f)}\right) \ . \tag{2.49}$$

If reheating occurs soon after inflation, the reheating temperature T_R is given by

$$\rho_{\rm v} = \frac{\pi^2 g_*(t_R) T_R^4}{30} \ , \tag{2.50}$$

where t_R the reheating time. After reheating entropy is conserved, so we have

$$\frac{2\pi^2 g_{S*}(t_R) T_R^3 a(t_R)^3}{45} = \frac{2\pi^2 g_{S*}(t_0) T_0^3 a(t_0)^3}{45} \ , \tag{2.51}$$

which leads to

$$\left(\frac{a(t_0)}{a(t_f)}\right) = \frac{g_{S*}(t_R)^{1/3} T_R}{g_{S*}(t_0)^{1/3} T_0} \simeq \frac{(100)^{1/3} 10^{16}\text{GeV}}{(43/11)^{1/3}\ 3\times 10^{-4}\text{eV}} \simeq 10^{29} \ , \tag{2.52}$$

where we have used $T_R = 10^{16}$ GeV and $g_{S*}(t_R) \simeq 100$. As for t_i we make a rough estimation as

$$t_i \sim H(t_i)^{-1} = \frac{\sqrt{3} M_G}{\rho_{\rm v}^{1/2}} \tag{2.53}$$

$$\sim \frac{\sqrt{3}\times 2.4\times 10^{18}\text{GeV}}{(\pi^2 g_*(t_R)/30)^{1/2}(10^{16}\text{GeV})^2} \sim 10^{-14}\text{GeV}^{-1} \sim 10^{-28}\text{cm} \ . \tag{2.54}$$

Here we have used Eq. (2.50). Thus, we obtain

$$L_{\rm H}(t_0) \sim 10 Z \text{ cm} \ . \tag{2.55}$$

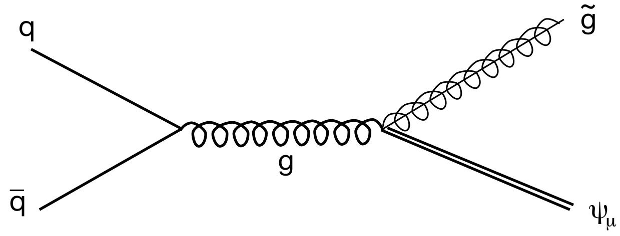

Figure 2.5.: Gravitino production.

On the other hand, taking into account that photons from the most distant universe we can see were emitted at the recombination epoch, the size of the observable universe is given by

$$L_{\text{obs}} = a(t_0)\int_{t_{\text{rec}}}^{t_0} \frac{dt}{a(t)} \sim t_0 \sim H_0^{-1} \sim 10^{28}\ \text{cm}\ . \tag{2.56}$$

If $L_{\text{obs}} < L_{\text{H}}$ the observable universe was once inside the horizon. This happens if sufficient duration of inflation, i.e. $N_{\text{tot}} = H_{\text{inf}}\Delta t > 62$ is realized. Thus, the horizon problem is solved by inflation.

2.2.3. Monopole and gravitino problems

The monopole problem is solved because monopoles existing before inflation are diluted by a factor $\exp(-3N) \lesssim 10^{-85}$. Therefore there are no monopoles in the present universe.

In the same way gravitinos existing before inflation are diluted away. However, the situation is a little complicated. Gravitinos are also produced thermally during reheating after inflation by, e.g. scattering of quarks $(q+\bar{q} \to \tilde{g}+\psi_\mu)$(Fig. 2.5). The number density of the secondarily produced gravitinos at reheating is estimated as

$$n_{3/2} \sim n_q^2 \sigma H^{-1} \sim 10^{-2} T_R^6 \frac{1}{M_G^2}\frac{M_G}{T_R^2} \sim 10^{-2}\frac{T_R^4}{M_G}\ , \tag{2.57}$$

where $n_q(\sim T_R^3)$ is the quark number density, $\sigma(\sim 10^{-2}/M_G^2)$ is the cross section, and T_R is the reheating temperature. So the gravitino-to-entropy ratio $n_{3/2}/s$ is given by

$$\frac{n_{3/2}}{s} \sim \frac{n_{3/2}}{10^2\, T_R^3} \sim 10^{-4}\frac{T_R}{M_G} \sim 10^{-12}\left(\frac{T_R}{10^{10}\text{GeV}}\right)\ . \tag{2.58}$$

For gravitinos with mass about 1 TeV, this leads to entropy production when they decay. At the decay time

$$\left.\frac{\rho_{3/2}}{\rho_\gamma}\right|_{\text{decay}} \sim \frac{m_{3/2}n_{3/2}}{T_{\text{decay}}n_\gamma} \sim 10^{-11}\frac{1\ \text{TeV}}{1\ \text{keV}}\left(\frac{T_R}{10^{10}\text{GeV}}\right)\ , \tag{2.59}$$

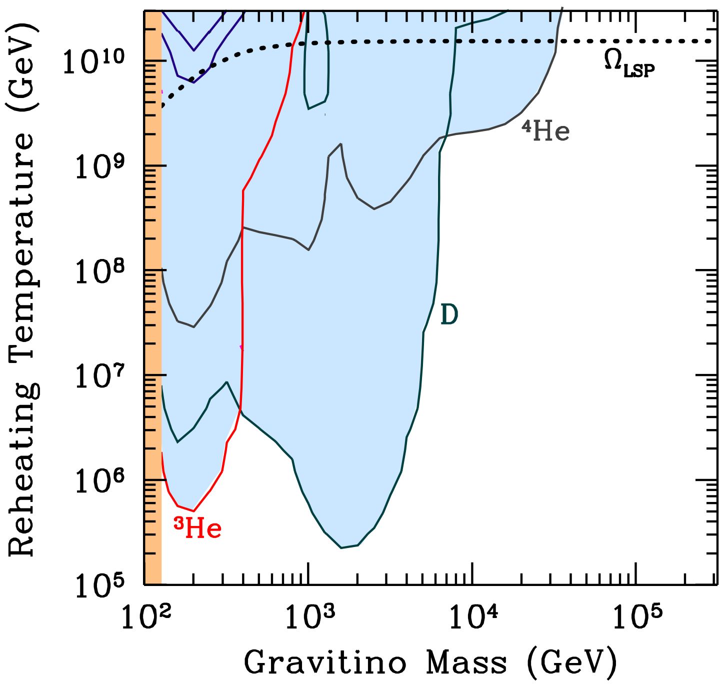

Figure 2.6.: Constraint on the reheating temperature from BBN.

where we have used $n_\gamma \simeq s/7$ at $T < m_e$. In order for gravtinos not to produce large entropy the reheating temperature T_R should be smaller than about 10^{12} GeV. Furthermore, energetic particles such as photon and gluons produced in the gravitino decay can destroy the light elements (^{4}He, ^{3}He and D) synthesized in BBN, from which we obtain the more stringent constraint on the reheating temperature,

$$T_R \lesssim 10^6 \text{ GeV} \quad \text{for } m_{3/2} \simeq 0.1 - 40 \text{ TeV} , \tag{2.60}$$

as seen in Fig. 2.6.

2.3. Chaotic Inflation

In this section, we consider a chaotic inflation model as a concrete model of inflation. The chaotic inflation model is the simplest among many inflation models proposed so

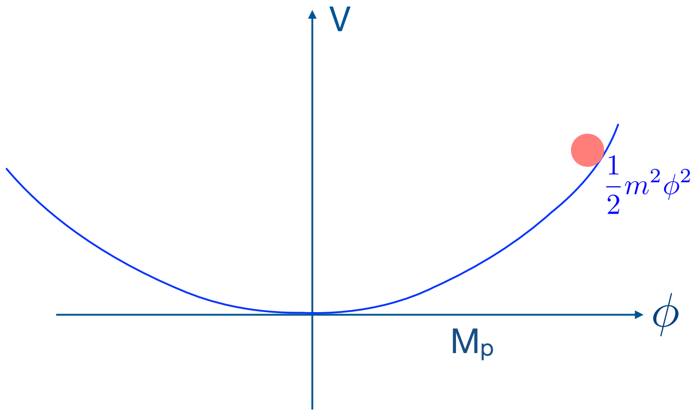

Figure 2.7.: Potential of chaotic inflation.

far and inflation takes place by one real scalar field with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) \tag{2.61}$$

$$V(\phi) = \frac{1}{2}m^2\phi^2 \ , \tag{2.62}$$

where m is the mass of the scalar field. The potetila $V(\phi)$ is shown in Fig. 2.7. In general, the potential V for chaotic inflation can be

$$V(\phi) = \frac{\lambda}{n}\frac{\phi^n}{M_G^{n-4}} \ , \tag{2.63}$$

where λ is the coupling constant. Hereafter, we call a scalar field which causes inflation inflaton.

2.3.1. Chaotic condition in the early universe

First, let us consider the initial condition at the Planck time. The Heisenberg uncertainty between energy and time implies

$$\Delta E \ \Delta t \gtrsim 1 \ . \tag{2.64}$$

Let us apply this relation to the inflaton field. The uncertainty of the energy density of the inflaton is estimated as

$$\Delta\rho \simeq \frac{\Delta E}{L^3} \gtrsim \frac{1}{\Delta t L^3} \ , \tag{2.65}$$

where L is the size of the region we consider, but it must be smaller than the horizon, so $L \lesssim \ell_{\rm pl} = M_G^{-1}$ ($\ell_{\rm pl}$: Planck length). In addition, Δt should be less than $t_{\rm pl} = M_G^{-1}$. Thus, we obtain

$$\boxed{\Delta\rho \sim M_G^4} \ . \tag{2.66}$$

This is called the chaotic condition in the early universe and means that the inflaton field at the Planck time has energy density of M_G^4.

From the Lagrangian (2.61) the energy momentum tensor is written as

$$T_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)}\partial_\nu\phi - g_{\mu\nu}\mathcal{L} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\mathcal{L} \ , \tag{2.67}$$

where the metric is $g_{\mu\nu} = \mathrm{diag}(1, -a^2, -a^2, -a^2)$. The energy density of the inflaton is then given by

$$\rho_\phi = T_{00} = \frac{1}{2}(\partial_0\phi)^2 + \frac{1}{2a^2}(\partial_i\phi)^2 + V(\phi) \ , \tag{2.68}$$

where the first, second and third terms are called kinetic, gradient and potential energies, respectively. (Notice that $\partial^i\phi\partial_i\phi = -(1/a^2)\partial_i\phi\partial_i\phi$ ....) From Eq. (2.66) we expect

$$(\partial_0\phi)^2 \sim M_G^4 \tag{2.69}$$

$$\frac{1}{a^2}(\partial_i\phi)^2 \sim M_G^4 \tag{2.70}$$

$$V(\phi) \sim M_G^4 \ . \tag{2.71}$$

In particular, for $V = m^2\phi^2/2$ the initial value of the inflaton, ϕ_i satisfies $m^2\phi_i^2 \sim M_G^4$ which leads to

$$\phi_i \sim \frac{M_G^2}{m} \gg M_G \qquad \text{for } m \ll M_G \ . \tag{2.72}$$

Thus, the initial value of the inflaton is much larger than the Planck scale at the Planck time.

2.3.2. Cosmological evolution of the inflaton

If the inflaton field ϕ dominates the energy density of the universe, from Eq.(2.68) the Friedman equation is written as

$$\left(\frac{\dot a}{a}\right)^2 = \frac{1}{3M_G^2}\left(\frac{1}{2}\dot\phi^2 + \frac{1}{2a^2}(\partial_i\phi)^2 + V(\phi)\right) \ . \tag{2.73}$$

On the other hand the equation of motion for ϕ is derived from the action

$$S = \int d^4x \, \sqrt{-g}\mathcal{L} = \int d^4x \; a^3\mathcal{L} \tag{2.74}$$

$$= \int d^4x \; a^3 \left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right) \ , \tag{2.75}$$

from which we obtain

$$\begin{aligned}\frac{\delta S}{\delta\phi} &= \partial_\mu\left(\frac{\partial(a^3\mathcal{L})}{\partial(\partial_\mu\phi)}\right) - \frac{\partial(a^3\mathcal{L})}{\partial\phi} \\ &= \partial_\mu(a^3 g^{\mu\nu}\partial_\nu\phi) + a^3\frac{dV}{d\phi} \\ &= \partial_0(a^3\partial_0\phi) - \partial_i(a^3 a^{-2}\partial_i\phi) + a^3\frac{dV}{d\phi} = 0\ .\end{aligned} \tag{2.76}$$

Thus, the equation of motion is given by

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} - \frac{1}{a^2}\triangle\phi = -\frac{dV}{d\phi}\ . \tag{2.77}$$

Let us suppose that there is a region with size $O(M_G^{-1})$ where $(\partial_\mu\phi)^2 < V(\phi)$ and assume that $\dot{\phi}^2 \ll V$ and $\ddot{\phi} \ll dV/d\phi$. Then, Eqs.(2.73) and (2.77) are

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3M_G^2}V(\phi) \tag{2.78}$$

$$3\frac{\dot{a}}{a}\dot{\phi} = -\frac{dV}{d\phi}\ . \tag{2.79}$$

Here we neglect terms with spatial derivatives because they decay as $1/a^2$. From Eq. (2.78)

$$\left(\frac{\dot{a}}{a}\right) = \frac{(V(\phi))^{1/2}}{\sqrt{3}M_G}\ , \tag{2.80}$$

which is substituted into Eq. (2.79),

$$3\dot{\phi}\frac{V^{1/2}}{\sqrt{3}M_G} = -\frac{dV}{d\phi}\ . \tag{2.81}$$

For the inflaton with potential $V(\phi) = \frac{1}{2}m^2\phi^2$, we obtain

$$\dot{\phi} = -\frac{\sqrt{2}}{\sqrt{3}}mM_G\ , \tag{2.82}$$

from which the solution is given by

$$\phi = \phi_i - \frac{\sqrt{2}}{\sqrt{3}}mM_G(t-t_i)\ , \tag{2.83}$$

with initial condition $\phi(t_i) = \phi_i$. Using this solution in Eq. (2.78), we obtain

$$\frac{\dot{a}}{a} = \frac{m}{\sqrt{6}M_G}\phi = \frac{m}{\sqrt{6}M_G}\left[\phi_i - \frac{\sqrt{2}}{\sqrt{3}}mM_G(t-t_i)\right]\ . \tag{2.84}$$

This can be easily integrated as

$$\ln\frac{a}{a_i} = \frac{m}{\sqrt{6}M_G}\left[\phi_i(t-t_i) - \frac{1}{\sqrt{6}}mM_G(t-t_i)^2\right] \tag{2.85}$$

$$= \frac{1}{4M_G^2}\left(\phi_i^2 - \phi^2\right) \ . \tag{2.86}$$

Finally we obtain

$$a = a_i \exp\left[\frac{1}{4M_G^2}\left(\phi_i^2 - \phi^2\right)\right] \ . \tag{2.87}$$

2.3.3. Slow roll condition

In solving dynamics of the inflaton we have assumed that $\dot{\phi}^2/2 \ll V$ and $|\ddot{\phi}| \ll |dV/d\phi|$. We now derive the condition for which these inequalities are satisfied. This condition is called slow roll condition. First let us consider the condition for $\dot{\phi}^2 \ll V$. From Eq. (2.79)

$$\dot{\phi} \simeq -\frac{V'}{3H} \qquad (V' \equiv dV/d\phi) \ , \tag{2.88}$$

which leads to

$$\frac{1}{2}\dot{\phi}^2 \simeq \frac{(V')^2}{18H^2} = \frac{(V')^2}{6V}M_G^2 \ll V \ . \tag{2.89}$$

Introducing the slow-roll parameter ϵ defined as

$$\boxed{\epsilon \equiv \frac{1}{2}\left(\frac{V'}{V}\right)^2 M_G^2} \ , \tag{2.90}$$

$\dot{\phi}^2/2 \ll V$ is satisfied if

$$\epsilon \ll 1 \ . \tag{2.91}$$

Nest let us consider $|\ddot{\phi}| \ll |dV/d\phi|$. Differentiating Eq. (2.88) with respect to t,

$$\ddot{\phi} \simeq -\frac{V''}{3H}\dot{\phi} + \frac{V'\dot{H}}{3H^2} \simeq \frac{V''V'}{9H^2} + \frac{V'\dot{H}}{3H^2} \ . \tag{2.92}$$

From Eq. (2.78),

$$2H\dot{H} \simeq \frac{V'}{3M_G^2}\dot{\phi} \simeq \frac{(V')^2}{9M_G^2 H} \ , \tag{2.93}$$

which leads to

$$\frac{\dot{H}}{H^2} \simeq \frac{(V')^2}{18M_G^2 H^4} = \frac{(V')^2}{2V^2}M_G^2 = \epsilon \ . \tag{2.94}$$

Using this relation, $\ddot{\phi}$ is written as

$$\ddot{\phi} \simeq V'\left(\frac{V''}{3V}M_G^2 + \epsilon\right) \ . \tag{2.95}$$

Therefore, $|\ddot{\phi}| \ll |dV/d\phi|$ is satisfied if

$$|\eta| \ll 1 \ , \tag{2.96}$$

where η is another slow-roll parameter defined as

$$\boxed{\eta \equiv \frac{V''}{V} M_G^2} \ . \tag{2.97}$$

When the slow-roll parameters ϵ and $|\eta|$ are much smaller than 1, the inflaton so slowly rolls down the potential that the potential energy is almost constant, which drives inflation. Conversely, inflation ends when $\epsilon \simeq 1$ or $|\eta| \simeq 1$.

So far we have derived the slow-roll condition for a generic inflaton potential. Let us calculate the slow-roll parameters for chaotic inflation with potential $V = m^2\phi^2/2$. ϵ and η are given by

$$\epsilon = \frac{1}{2}\left(\frac{m^2\phi^2}{m^2\phi^2/2}\right)^2 M_G^2 = \frac{2M_G^2}{\phi^2} \tag{2.98}$$

$$\eta = \frac{m^2}{m^2\phi^2/2} M_G^2 = \frac{2M_G^2}{\phi^2} \ . \tag{2.99}$$

Both slow-roll parameters are much smaller than 1 when $\phi \gg \sqrt{2}M_G$. This is perfectly consistent with the initial chaotic condition which predicts the initial value of the inflaton ϕ_i as

$$V = \frac{1}{2}m^2\phi_i^2 \sim M_G^4 \qquad \Rightarrow \qquad \phi_i \sim \left(\frac{M_G}{m}\right) M_G \gg M_G \quad \text{for } m \ll M_G \ . \tag{2.100}$$

Thus, chaotic inflation naturally occurs.

As mentioned above, inflation end when $\epsilon \simeq 1$ or $|\eta| \simeq 1$. In the case of chaotic inflation with potential $V = m^2\phi^2/2$, inflation ends for $\phi = \phi_f \simeq \sqrt{2}M_G$. From Eq. (2.87), during inflation the scale factor increases by

$$\frac{a_f}{a_i} = \exp\left[\frac{1}{4M_G^2}(\phi_i^2 - 2M_G^2)\right] \sim \exp\left[\left(\frac{M_G}{m}\right)^2\right] \ . \tag{2.101}$$

Later we will see that $m \simeq 10^{13}$ GeV which leads to $a_f/a_i \sim \exp(10^{10})$. This is enough to solve flatness and horizon problems.[2]

[2]Actually, Eq. (2.87) cannot be used for $\phi \gtrsim (M_G/m)^{1/2}M_G$ because fluctuations of the inflaton field produced by inflation cannot be neglected in the inflaton dynamics. Even taking this into account, we have sufficient inflation with $a_f/a_i \sim \exp(M_G/m) \sim \exp(10^5)$.

2.4. Slow-roll inflation

In the previous section we have seen that chaotic inflation naturally takes place and provide sufficient quasi-exponential expansion of the universe which solves the problems of the standard cosmology. The dynamics of the inflaton field is solved by using the slow-roll approximation which is valid if slow-roll parameters ϵ and η are much smaller than unity (slow-roll condition). When the slow-roll condition is satisfied, the inflaton field slowly rolls down the potential and hence the potential energy effectively behaves as cosmological constant by which inflation takes place. In fact, in almost all inflation models proposed so far inflation occurs when the slow-roll condition is satisfied. Therefore this type of inflation is called slow-roll inflation. In this section we consider the slow-roll inflation without specifying concrete form of the inflaton potential.

2.4.1. Accelerated expansion

In this subsection we derive the condition for accelerated expansion of the universe. The acceleration of the universe is given by [Eq. (1.49)]

$$\ddot{a} = -\frac{1}{6M_G^2}(\rho + 3P)a \ . \tag{2.102}$$

From the energy-momentum tensor for the homogeneous inflaton field [see, Eq. (2.67)], the energy density ρ and pressure P are given by

$$\rho = \frac{1}{2}\dot{\phi}^2 + V \ , \tag{2.103}$$

$$P = \frac{1}{2}\dot{\phi}^2 - V \ , \tag{2.104}$$

where we have used $T_{00} = \rho$ and $T_{ij} = -Pg_{ij}$. Using Eqs. (2.103) and (2.104) we obtain

$$\ddot{a} = \frac{1}{3M_G^2}(V - \dot{\phi}^2)a \ . \tag{2.105}$$

Thus, the accelerated expansion ($\ddot{a} > 0$) is realized if $V > \dot{\phi}^2$. Using the slow-roll approximation $\dot{\phi} \simeq -V'/(3H)$, this condition is written as

$$1 > \frac{\dot{\phi}^2}{V} = \frac{(V')^2}{9H^2V} = \frac{1}{3}\left(\frac{V'}{V}\right)^2 M_G^2 \simeq \epsilon \ . \tag{2.106}$$

Therefore, the accelerated expansion takes place when the slow-roll parameter ϵ satisfies $\epsilon \gtrsim 1$. For $\epsilon \ll 1$ the kinetic energy is negligible ($\dot{\phi}^2 \ll V$), which leads to $P = V = -\rho$, so the inflaton potential behaves as cosmological constant.

2.4.2. e-fold N

The e-fold N is defined by

$$N = \ln \frac{a(t_f)}{a(t_N)} \ . \tag{2.107}$$

So e^N represents how the scale factor increases from $t = t_N$ to the end of inflation ($t = t_f$). From Eq. (2.107)

$$N = \int_{a(t_N)}^{a(t_f)} da \, \frac{1}{a} = -\int_{t_f}^{t_N} dt \, \frac{da}{dt}\frac{1}{a} = -\int_{t_f}^{t_N} dt \, H = -\int_{\phi_f}^{\phi_N} d\phi \, \frac{dt}{d\phi} H \ , \tag{2.108}$$

where $\phi_N = \phi(t_N)$. Using Eq. (2.88),

$$N = \int_{\phi_f}^{\phi_N} d\phi \, \frac{3H^2}{V'} = \int_{\phi_f}^{\phi_N} d\phi \, \frac{V}{V' M_G^2} \ , \tag{2.109}$$

where we have used $H^2 = V/(3M_G^2)$ during inflation. Finally, using the slow-roll parameter e-fold N is given by

$$\boxed{N = \int_{\phi_f}^{\phi_N} d\phi \, \frac{V}{V' M_G^2} = \int_{\phi_f}^{\phi_N} d\phi \, \frac{1}{\sqrt{2\epsilon} M_G}} \tag{2.110}$$

If we take the initial value of the inflaton in place of ϕ_N we obtain the total e-fold $N_{\rm tot}$ as

$$N_{\rm tot} = \int_{\phi_f}^{\phi_i} d\phi \, \frac{V}{V' M_G^2} \tag{2.111}$$

For example, for chaotic inflation with $V = m^2\phi^2/2$, Eq. (2.110) gives

$$N = \int_{\phi_f}^{\phi_N} d\phi \, \frac{m^2\phi^2}{m^2 \phi M_G^2} = \frac{1}{4M_G^2}(\phi_N^2 - \phi_f^2) \ , \tag{2.112}$$

which is the same result directly calculated using Eqs. (2.107) and (2.87).

2.4.3. After inflation

Here let us see what happens after inflation. The equation of motion for the inflaton field is given by

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + V' = 0 \ . \tag{2.113}$$

We assume that after inflation the potential is dominated by the quadratic term, i.e. $V \simeq m^2\phi^2/2$. Then the equation of motion is written as

$$\ddot{\phi} + 2\frac{\dot{a}}{a}\dot{\phi} + m^2\phi = 0 \ . \tag{2.114}$$

Soon after inflation $\dot{a}/a = H$ becomes much smaller than m ($H \ll m$), so as zero-th approximation we can neglect the second term and obtain the equation for a simple oscillator as

$$\ddot{\phi} \simeq -m^2\phi \ , \tag{2.115}$$

which has a solution

$$\phi = \Phi\cos(mt + B) \ , \tag{2.116}$$

where Φ is the amplitude of the oscillation ($\sim \phi_f$) and B is a constant determined by $\phi(t_f) = \phi_f$.

Next, let us take into account the cosmic expansion. Multiplying Eq. (2.113) by $\dot{\phi}$,

$$\ddot{\phi}\dot{\phi} + V'\dot{\phi} = -3\frac{\dot{a}}{a}\dot{\phi}^2 \quad \Rightarrow \quad \left(\frac{1}{2}\dot{\phi}^2 + V\right)^{\cdot} = -3\frac{\dot{a}}{a}\dot{\phi}^2 \ , \tag{2.117}$$

where $\dot{\phi}^2/2 + V$ is the energy density of the inflaton field ρ_ϕ. Since the time scale of the inflaton oscillation ($\sim m^{-1}$) is much shorter than the expansion time scale ($\sim H^{-1}$), we can replace fast oscillating terms by their average over an oscillation period, which leads to

$$\dot{\rho}_\phi = -3\frac{\dot{a}}{a}\langle\dot{\phi}^2\rangle \ , \tag{2.118}$$

Using Eq. (2.116) ρ_ϕ and $\langle\dot{\phi}^2\rangle$ are

$$\rho_\phi = \frac{1}{2}m^2\Phi^2\cos^2(mt + B) + \frac{1}{2}m^2\Phi^2\sin^2(mt + B) = \frac{1}{2}\Phi^2 m^2 \tag{2.119}$$

$$\langle\dot{\phi}^2\rangle = \langle m^2\Phi^2\sin^2(mt + B)\rangle = \frac{1}{2}\Phi^2 m^2 = \rho_\phi \ . \tag{2.120}$$

Thus, with effect of the cosmic expansion, the energy density of the inflaton oscillation is given by

$$\dot{\rho}_\phi = -3\frac{\dot{a}}{a}\rho_\phi \ , \tag{2.121}$$

which leads to

$$\boxed{\rho_\phi \propto a^{-3}} \ . \tag{2.122}$$

So the inflaton oscillation behaves like matter. From Eq. (2.122) the oscillation amplitude of the inflaton decreases as $\Phi \propto a^{-3/2}$.

2.4.4. Reheating

The inflaton oscillation lasts until it decays through couplings with other particles. Thus, inflaton decays into other particles whose successive scatterings and decays form thermal

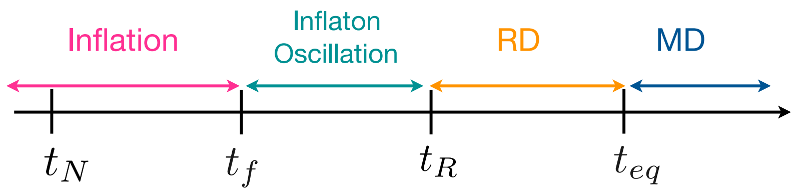

Figure 2.8.: Timeline of inflationary universe.

plasma with temperature T_R. T_R is called reheating temperature. If we assume that thermalization takes place immediately when the inflaton decays, the reheating temperature is estimated from $H(T_R) = \Gamma_\phi$ where Γ_ϕ is the decay rate. This just means that the decay occurs when the cosmic time ($\sim H^{-1}$) equals lifetime of the inflaton ($= \Gamma^{-1}$). The Hubble parameter H when the universe is dominated by the thermal radiation with T_R is given by

$$H(T_R) = \frac{1}{M_G}\left(\frac{g_*\pi^2 T_R^4}{90}\right)^{1/2} , \tag{2.123}$$

which is equal to Γ_ϕ. So the reheating temperature is estimated as

$$\boxed{T_R = \left(\frac{90}{g_*\pi^2}\right)^{1/4} \sqrt{\Gamma_\phi M_G}} . \tag{2.124}$$

2.4.5. Cosmological scale and e-fold

Let us derive the relation between the present cosmological scale L and the Hubble radius $H(t_N)^{-1}$ at $t = t_N$. The Hubble radius during inflation is often called "horizon" beacuse the exponentially expanding universe has the event horizon equal to H^{-1}. (Strictly speaking, however, there is no event horizon because inflation ends in a finite time.) The scale $H(t_N)^{-1}$ is stretched by the cosmic expansion to the present scale given by $[a(t_0)/a(t_N)]H(t_N)^{-1}$. From t_N to t_0 the universe experiences several stages of the cosmic expansion as shown in Fig. 2.8. Correspondingly L is written as

$$L = \frac{a(t_0)}{a(t_N)} H(t_N)^{-1} = \frac{a(t_f)}{a(t_N)}\frac{a(t_R)}{a(t_f)}\frac{a(t_0)}{a(t_R)} H(t_N)^{-1} , \tag{2.125}$$

where t_f is the end of inflation and t_R is the time of reheating. First, we assume that the Hubble parameter is almost constant during inflation and is given by H_I, so

$H(t_N)^{-1} = H_I^{-1}$. From the definition of the e-fold,

$$\frac{a(t_f)}{a(t_N)} = e^N \ . \tag{2.126}$$

From t_f to t_R, the universe is dominated by the oscillation energy of the inflaton which decreases as a^{-3}. Thus, just before reheating the energy density ρ_ϕ is given by

$$\rho_\phi(t_R) = \left(\frac{a(t_f)}{a(t_R)}\right)^3 \rho_\phi(t_f) \simeq \left(\frac{a(t_f)}{a(t_R)}\right)^3 3H_I^2 M_G^2 \ , \tag{2.127}$$

where we have used $H(t_f)^2 = \rho_\phi(t_f)/(3M_G^2) \simeq H_I^2$. If reheating is instantaneous,

$$\rho_\phi(t_R) \simeq \frac{\pi^2}{30} g_* T_R^4 \ . \tag{2.128}$$

Thus, we obtain

$$\frac{a(t_R)}{a(t_f)} \simeq \left(\frac{90 M_G^2 H_I^2}{\pi^2 g_* T_R^4}\right)^{1/3} \ . \tag{2.129}$$

After reheating, the universe expands adiabatically and the entropy is conserved. This leads to

$$s(T_R) a(t_R)^3 = s_0 a(t_0)^3 \ , \tag{2.130}$$

from which we obtain

$$\frac{a(t_0)}{a(t_R)} = \left(\frac{s(T_R)}{s_0}\right)^{1/3} = \left(\frac{2\pi^2 g_* T_R^3}{45 s_0}\right)^{1/3} \ . \tag{2.131}$$

Therefore, the relation between L and N is written as

$$L = e^N \left(\frac{90 M_G^2 H_I^2}{\pi^2 g_* T_R^4}\right)^{1/3} \left(\frac{2\pi^2 g_* T_R^3}{45 s_0}\right)^{1/3} H_I^{-1} \ . \tag{2.132}$$

Using $s_0 \simeq 2.8 \times 10^3 \ \mathrm{cm}^{-3}$, finally we obtain

$$\boxed{N = 52.6 + \ln\left(\frac{L}{1000\mathrm{Mpc}}\right) + \frac{1}{3}\ln\left(\frac{T_R}{10^{10}\mathrm{GeV}}\right) + \frac{1}{3}\ln\left(\frac{H_I}{10^{10}\mathrm{GeV}}\right)} \ . \tag{2.133}$$

If $H(t_N) \neq H(t_f)$ the above relation is changed to

$$N = 52.6 + \ln\left(\frac{L}{1000\mathrm{Mpc}}\right) + \frac{1}{3}\ln\left(\frac{T_R}{10^{10}\mathrm{GeV}}\right) \tag{2.134}$$

$$+ \ln\left(\frac{H(t_N)}{10^{10}\mathrm{GeV}}\right) - \frac{2}{3}\ln\left(\frac{H(t_f)}{10^{10}\mathrm{GeV}}\right) \ . \tag{2.135}$$

3. Generation and Evolution of Density Fluctuations

3.1. Generation of density fluctuations

3.1.1. Fluctuations of an inflaton field during inflation

Let us consider quantum fluctuations of an inflaton field. In slow-roll inflation the effective mass during inflation, which is given by $m_{\rm eff}^2 = V'' = 3\eta H_{\rm inf}^2$, is much small than the Hubble parameter, so for the moment we assume that an inflaton is mass less. Furthermore, for simplicity, we assume that the Hubble parameter during inflation is constant. The equation of motion for the inflaton is written as

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} - \frac{1}{a^2}\Delta\phi = 0 \ , \tag{3.1}$$

where $a = \exp(Ht)$. The quantum scalar field ϕ is written as

$$\phi(t,\vec{x}) = \frac{1}{(2\pi)^{3/2}}\int d^3k \left[a_k\psi_k(t)e^{i\vec{k}\cdot\vec{x}} + a_k^\dagger\psi_k^*(t)e^{-i\vec{k}\cdot\vec{x}}\right] \ , \tag{3.2}$$

where a_k and $a_k^\dagger$ are annihilation and creation operators satisfying

$$[a_k, a_k^\dagger] = \delta^{(3)}(\vec{k} - \vec{q}) \ , \tag{3.3}$$

and $\psi_k(t)$ is the mode function. In Minkowski space we have

$$\phi(t,\vec{x}) = \frac{1}{(2\pi)^{3/2}}\int \frac{d^3k}{\sqrt{2k_0}} \left[a_k e^{-ik_0t + i\vec{k}\cdot\vec{x}} + a_k^\dagger e^{ik_0t - i\vec{k}\cdot\vec{x}}\right] \ . \tag{3.4}$$

So the mode function is given by $\psi_k(t) = 1/\sqrt{2k_0}e^{-ik_0t}$ in Minkowski space. The Lagrangian for the inflaton is given by

$$\mathcal{L} = a^3(\partial_\mu\phi\,\partial^\mu\phi - V(\phi)) \ , \tag{3.5}$$

from which the canonical momentum is derived as

$$\pi = \frac{\delta\mathcal{L}}{\delta\dot{\phi}} = a^3\dot{\phi} \ . \tag{3.6}$$

The canonical quantization requires

$$[\phi(t,\vec{x}), \pi(t,\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y}) \ . \tag{3.7}$$

The above commutation is calculated using Eq. (3.2) as

$$\begin{aligned}
[\phi(t,\vec{x}), \pi(t,\vec{y})] &= \frac{a^3}{(2\pi)^3}\int d^3k d^3q \left[a_k^\dagger \psi_k^* e^{-i\vec{k}\cdot\vec{x}} + a_k\psi_k e^{i\vec{k}\cdot\vec{x}},\ a_q^\dagger \dot{\psi}_q^* e^{-i\vec{q}\cdot\vec{y}} + a_q\dot{\psi}_q e^{i\vec{q}\cdot\vec{y}}\right] \\
&= \frac{a^3}{(2\pi)^3}\int d^3k d^3q \left\{[a_k^\dagger, a_q]\psi_k^*\dot{\psi}_q e^{-i(\vec{k}\cdot\vec{x}-\vec{q}\cdot\vec{y})} + [a_k, a_q^\dagger]\psi_k\dot{\psi}_q^* e^{i(\vec{k}\cdot\vec{x}-\vec{q}\cdot\vec{y})}\right. \\
&\quad \left. +[a_k^\dagger, a_q^\dagger]\psi_k^*\dot{\psi}_q^* e^{-i(\vec{k}\cdot\vec{x}+\vec{q}\cdot\vec{y})} + [a_k, a_q]\psi_k\dot{\psi}_q e^{i(\vec{k}\cdot\vec{x}+\vec{q}\cdot\vec{y})}\right\} \\
&= \frac{a^3}{(2\pi)^3}\int d^3k (\psi_k\dot{\psi}_k^* - \dot{\psi}_k\psi_k^*) e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \ ,
\end{aligned} \tag{3.8}$$

which should satisfy Eq. (3.7) and hence we obtain the normalization of the mode function,

$$\psi_k\dot{\psi}_k^* - \dot{\psi}_k\psi_k^* = \frac{i}{a^3} \ . \tag{3.9}$$

From equation of motion (3.1) the mode function ψ_k satisfies

$$\ddot{\psi}_k + 3H\dot{\psi}_k + k^2 e^{-2Ht}\psi_k = 0 \ . \tag{3.10}$$

Here let us use the conformal time τ instead of the usual time t. The conformal time is defined by $d\tau = dt/a(t) = e^{-Ht}dt$, so

$$\tau = -H^{-1}e^{-Ht} = -\frac{1}{aH} \ . \tag{3.11}$$

The conformal time τ changes from $-1/H$ to 0 as t changes from 0 to ∞. Using τ and redefining ψ as $\tau^{3/2}u$, Eq. (3.10) is rewritten as

$$u'' + \frac{1}{\tau}u' + \left(k^2 - \frac{9}{4\tau^2}\right)u = 0 \ , \tag{3.12}$$

where $' \equiv d/d\tau$. This is the Bessel differential equation and its solution is given by

$$\psi_k(t) = \frac{\sqrt{\pi}}{2}H\tau^{3/2}\left[C_1(k)H_{3/2}^{(1)}(k\tau) + C_2(k)H_{3/2}^{(2)}(k\tau)\right] \ , \tag{3.13}$$

where $H_{3/2}^{(1)}$ and $H_{3/2}^{(2)}$ are the Hankel functions,

$$H_{3/2}^{(2)}(x) = (H_{3/2}^{(1)}(x))^* = -\sqrt{\frac{2}{\pi x}}e^{-ix}\left(1+\frac{1}{ix}\right) \ . \tag{3.14}$$

Now we have to determine the integration constants C_1 and C_2. Here we adopt the following principle: quantization in de-Sitter space (=exponentially expanding space) should be the same as that in Minkowski space in the limit of $k \to \infty$. That means that at small scales ($k \to \infty$) the quantum field is not affected by the cosmic expansion. In Minkowski space, the mode function is given by $1/\sqrt{2k}\exp(-ikt) = 1/\sqrt{2k}\exp(-i\int kdt)$, so the mode function ψ should be

$$\psi_k = \frac{1}{a\sqrt{2k}} e^{-i\int \frac{k}{a}dt} = \frac{1}{a\sqrt{2k}} e^{-ik\tau} \qquad (k \to \infty) \ . \tag{3.15}$$

Here we have taken the normalization (3.9) into account. On the other hand the solution of the mode function Eq. (3.13) is written in the $k \to \infty$ limit as

$$\begin{aligned} \psi_k \ \longrightarrow \ & \frac{\sqrt{\pi}}{2} H\tau^{3/2} \left(-\sqrt{\frac{2}{\pi k\tau}}\right) \left[C_1(k)e^{ik\tau} + C_2(k)e^{-ik\tau}\right] \\ & = \frac{-H\tau}{\sqrt{2k}} \left[C_1(k)e^{ik\tau} + C_2(k)e^{-ik\tau}\right] \\ & = \frac{1}{a\sqrt{2k}} \left[C_1(k)e^{ik\tau} + C_2(k)e^{-ik\tau}\right] \ . \end{aligned} \tag{3.16}$$

Comparing with Eq.(3.15) we obtain

$$C_1(k) \ \to \ 0 \qquad\qquad C_2(k) \ \to \ 1 \ . \tag{3.17}$$

This only applies to any mode functions whose wavenumber k is much larger than the Hubble parameter at the beginning of the universe. As seen later relevant fluctuations which are responsible for the structure of the universe have such large wavenumber. Thus, for the mode functions with k we are interested in, we can set $C_2 = 1$ and $C_1 = 0$ and we obtain

$$\psi_k(t) = \frac{iH}{k\sqrt{2k}} \left(-\frac{ike^{-Ht}}{H} + 1\right) \exp\left(\frac{ik}{H}e^{-Ht}\right) = \frac{iH}{k\sqrt{2k}} \left(-\frac{ik}{aH} + 1\right) \exp\left(\frac{ik}{aH}\right) \ . \tag{3.18}$$

As the physical wavelength a/k becomes larger than the Hubble radius H^{-1}, the mode function Eq. (3.18) is given by

$$\psi_k(t) \ \to \ \frac{iH}{k\sqrt{2k}} \ , \tag{3.19}$$

which no longer oscillates and hence the scalar field behaves as classical one.

Hereafter we explicitly decomposes the scalar field ϕ into a homegeneous part and its fluctuation as

$$\phi(t, \vec{x}) = \bar{\phi}(t) + \delta\phi(t, \vec{x}) \ . \tag{3.20}$$

Then $\langle\delta\phi^2\rangle$ is calculated as

$$\begin{aligned}\langle\delta\phi^2\rangle = \int |\psi_k|^2 d^3k &= \frac{1}{(2\pi)^3}\int d\ln k\left(\frac{1}{2ka^2}+\frac{H^2}{2k^3}\right)\\ &= \frac{1}{2\pi^2}\int d^3k\left(\frac{k^2}{2a^2}+\frac{H^2}{2}\right)\\ &\simeq \frac{H^2}{(2\pi)^2}\int d\ln k \ ,\end{aligned} \tag{3.21}$$

where we have used $k/a \ll H$ in the last line. Therefore, the fluctuation of the inflaton is given by

$$\boxed{\delta\phi \simeq \frac{H}{2\pi}} \ . \tag{3.22}$$

Now we can regard the inflaton fluctuation as classical and express it as Fourier integral,

$$\delta\phi(\vec{x}) = \frac{1}{(2\pi)^3}\int d^3k\delta\phi_{\vec{k}}\, e^{i\vec{k}\cdot\vec{x}} \ . \tag{3.23}$$

$\langle(\delta\phi(\vec{x}))^2\rangle$ is calculated as

$$\begin{aligned}\langle(\delta\phi(\vec{x}))^2\rangle &= \frac{1}{(2\pi)^6}\int d^3kd^3k'\langle\delta\phi_{\vec{k}}\delta\phi_{\vec{k}'}\rangle e^{i\vec{x}\cdot(\vec{k}+\vec{k}')}\\ &= \frac{1}{(2\pi)^3}\int d^3k\frac{H^2}{2k^3} \ .\end{aligned} \tag{3.24}$$

So we obtain

$$\boxed{\langle\delta\phi_{\vec{k}}\delta\phi_{\vec{k}'}\rangle = (2\pi)^3\delta(\vec{k}+\vec{k}')\frac{H^2}{2k^3}} \ . \tag{3.25}$$

We introduce the power spectrum of scalar fluctuations $\mathcal{P}_{\delta\phi}$ which is defined by

$$\frac{4\pi k^3}{(2\pi)^6}\langle\delta\phi_{\vec{k}}\delta\phi_{\vec{k}'}\rangle = \delta(\vec{k}+\vec{k}')\mathcal{P}_{\delta\phi}(k) \ , \tag{3.26}$$

so $\mathcal{P}_{\delta\phi}$ is given by

$$\boxed{\mathcal{P}_{\delta\phi}(k) = \frac{H^2}{4\pi^2}} \ . \tag{3.27}$$

3.1.2. Effect of inflaton mass

Let us consider the effect of the mass term on the fluctuations of the inflaton fields. The equation of motion for the fluctuation of the inflaton field is written as

$$\delta\ddot{\phi} + 3\frac{\dot{a}}{a}\delta\dot{\phi} - \frac{1}{a^2}\Delta\delta\phi + m^2\delta\phi = 0 \ . \tag{3.28}$$

So the mode function ψ_k satisfies

$$\ddot{\psi}_k + 3H\dot{\psi}_k + \left(k^2 e^{-2Ht} + m^2\right)\psi_k = 0 \ , \tag{3.29}$$

which is written using $\tau = -1/(aH)$ and $\psi = \tau^{3/2} u$ as

$$u'' + \frac{1}{\tau}u' + \left(k^2 - \frac{\frac{9}{4} - \frac{m^2}{H^2}}{\tau^2}\right) u = 0 \ . \tag{3.30}$$

The generic solution for ψ_k is given by

$$\psi_k(t) = e^{i(\nu-3/2)\frac{\pi}{2}} \frac{\sqrt{\pi}}{2} H\tau^{3/2} \left[C_1(k)H_\nu^{(1)}(k\tau) + C_2(k)H_\nu^{(2)}(k\tau)\right] \ , \tag{3.31}$$

where ν is given by

$$\nu = \sqrt{\frac{9}{4} - \frac{m^2}{H^2}} \ . \tag{3.32}$$

Since $m \ll H$ during inflation ν is approximately written as

$$\nu \simeq \frac{3}{2} - \frac{m^2}{3H^2} \ . \tag{3.33}$$

In the same way as the massless case $\psi_k = \frac{1}{a\sqrt{2k}} e^{-ik\tau}$ for large k. The Hankel function $H_\nu(z)$ has the following form at large z:

$$H_\nu^{(2)}(z) = (H_\nu^{(1)}(z))^* \sim -\sqrt{\frac{2}{\pi z}} e^{-iz} \ , \tag{3.34}$$

which leads to $C_2 = 1$ and $C_1 = 0$. Thus, we obtain

$$\psi_k(t) = e^{i(\nu-3/2)\frac{\pi}{2}} \frac{\sqrt{\pi}}{2} H\tau^{3/2} H_\nu^{(2)}(k\tau) \ . \tag{3.35}$$

At the long wavelength limit $(k/a)^{-1} \gg H^{-1}$ (small $k\tau$ limit), the Hankel function is written as

$$\begin{aligned} H_\nu^{(2)}(k\tau) &\simeq i\frac{2^\nu \Gamma(\nu)}{\pi}(k\tau)^{-\nu} \\ &\simeq i\frac{\sqrt{2}}{\sqrt{\pi}}(k\tau)^{-3/2+\frac{m^2}{3H^2}} = i\frac{\sqrt{2}}{\sqrt{\pi}}(k\tau)^{-3/2}\left(-\frac{k}{aH}\right)^{\frac{m^2}{3H^2}} \ , \end{aligned} \tag{3.36}$$

where $\Gamma(\nu)$ is the gamma function and $\Gamma(3/2) = \sqrt{\pi}/2$. Then, after the wavelength exceeds the Hubble radius the mode function ψ_k is given by

$$\psi_k \simeq \frac{iH}{k\sqrt{2k}}\left(\frac{k}{aH}\right)^{\frac{m^2}{3H^2}} \ . \tag{3.37}$$

Therefore, for the case of an inflaton with mass m, the power spectrum of the inflaton fluctuations is given by

$$\boxed{\mathcal{P}_{\delta\phi}(k) = \frac{H^2}{4\pi^2}\left(\frac{k}{aH}\right)^{\frac{2m^2}{3H^2}}} \ . \tag{3.38}$$

3.2. Evolution of density fluctuations

So far, we have studied the fluctuations of the inflaton field without taking metric perturbations into account. Since fluctuations of the inflaton field affect metrics through Einstein equations and induce the metric perturbations. Moreover the metric perturbations are closely related to the density perturbations, from which the large scale structure of the universe is formed. Therefore, it is crucial to understand the evolution of the metric perturbations. Here is a remark. The evolution of the inflaton fluctuations are also affected by metric perturbations. However, it is found that the back-reaction on the inflaton fluctuations from metric perturbations is negligible for some time after wavelengths of the scalar fluctuations exceed the Hubble radius. Thus, the calculation in the previous section is justified.

3.2.1. Metric perturbations (scalar)

Let us introduce the metric perturbations,

$$ds^2 = a^2(1+\phi)d\tau^2 - 2a^2\omega_{,j}d\tau dx^j - a^2[\delta_{ij}(1+2\psi) + 2\chi_{,ij}]dx^i dx^j \ , \tag{3.39}$$

where $\chi_{ij} = (\partial_i\partial_j - \delta_{ij}\Delta/3)\chi$. Here we have used the conformal time τ instead of the usual cosmic time t. Since we only consider linear perturbations, it is convenient to study Fourier modes of perturbation quantities. So ϕ is written as

$$\phi(\vec{x}) = \sum_{\vec{k}} \phi(\vec{k})e^{i\vec{k}\cdot\vec{x}} = \sum_{\vec{k}} \phi(\vec{k})Q(\vec{k},\vec{x}) \equiv \phi Q \ , \tag{3.40}$$

where $Q(\vec{k},\vec{x}) = e^{i\vec{k}\cdot\vec{x}}$. We also introduce Q_i and Q_{ij} defined as

$$Q_i \equiv -\frac{1}{k}Q_{,i} = -\frac{ik_i}{k}Q \tag{3.41}$$

$$Q_{ij} \equiv \frac{1}{k^2}Q_{,ij} + \frac{1}{3}\delta_{ij}Q \ . \tag{3.42}$$

If we define the Fourier mode of ω and χ as

$$\omega(\vec{x}) = \sum_{\vec{k}} k^{-1}\omega(\vec{k})e^{i\vec{k}\cdot\vec{x}} \ , \tag{3.43}$$

$$\chi(\vec{x}) = \sum_{\vec{k}} k^{-2}\chi(\vec{k})e^{i\vec{k}\cdot\vec{x}} \ , \tag{3.44}$$

then

$$\omega_{,j}(\vec{x}) = \partial_j \sum_{\vec{k}} k^{-1}\omega(\vec{k})Q(\vec{k},\vec{x}) = -\sum_{\vec{k}} \omega(\vec{k})Q_j(\vec{k},\vec{x}) \equiv -\omega Q_j \ . \tag{3.45}$$

and

$$\chi_{ij} = \sum_{\vec{k}} k^{-2}\chi(\vec{k})\left(Q_{,ij} + \frac{k^2}{3}\delta_{ij}Q\right) = \sum_{\vec{k}} \chi(\vec{k})Q_{ij}(\vec{k},\vec{x}) \equiv \chi Q_{ij} \ . \tag{3.46}$$

Therefore, the metrics are rewritten as

$$g_{00} = a^2[1 + 2\phi Q] \tag{3.47}$$

$$g_{0j} = a^2\omega Q_j \tag{3.48}$$

$$g_{ij} = -a^2[\delta_{ij}(1 + 2\psi Q) + 2\chi Q_{ij}] \ . \tag{3.49}$$

The definition of metric perturbations is not unique since we are allowed to make coordinate transformations in general relativity. The ambiguity in the metric perturbations is called gauge freedom. Let us consider the following (scalar) coordinate transformation:

$$\tilde{\tau} = \tau + TQ \tag{3.50}$$

$$\tilde{x}^j = x^j + LQ^j \ . \tag{3.51}$$

The above transformation changes the metric $g_{\mu\nu}$ as

$$\tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta}(x) \ , \tag{3.52}$$

which leads to

$$\begin{aligned} \tilde{g}_{\mu\nu}(\tau, x^i) &= \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta}(\tau - TQ, x^i - LQ^i) \\ &\simeq g_{\mu\nu}(\tau, x^i) - g_{\alpha\nu}(\tau, x^i)\delta x^\alpha_{,\mu} - g_{\alpha\mu}(\tau, x^i)\delta x^\alpha_{,\nu} - g_{\mu\nu,\alpha}(\tau, x^i)\delta x^\alpha \ . \end{aligned} \tag{3.53}$$

Thus, g_{00} changes as

$$\begin{aligned} \tilde{g}_{00} - g_{00} &= -2g_{\alpha 0}\delta x^\alpha_{,0} - g_{00,\alpha}\delta x^\alpha \\ &= -2g_{00}\delta x^0_{,0} - g_{00,0}\delta x^0 \\ &= -2a^2T'Q - 2aa'TQ \\ &= a^2[-2T'Q - 2\mathcal{H}TQ] \ , \end{aligned} \tag{3.54}$$

where $\mathcal{H} = a'/a$. g_{0j} is transformed as

$$\begin{aligned} \tilde{g}_{0j} - g_{0j} - g_{\alpha j}\delta x^\alpha_{,0} - g_{\alpha 0}\delta x^\alpha_{,j} - g_{0j,\alpha}\delta x^\alpha \\ = -g_{ij}\delta x^i_{,0} - g_{00}\delta x^0_{,j} \\ = a^2\delta_{ij}L'Q^i + a^2TkQ_j \\ = a^2[L' + kT]Q_j \ . \end{aligned} \tag{3.55}$$

Therefore, the metric perturbations are transformed as

$$\tilde{\phi} = \phi - T' - \mathcal{H}T \tag{3.56}$$

$$\tilde{\omega} = \omega + L' + kT \tag{3.57}$$

$$\tilde{\psi} = \psi - \frac{kL}{3} - \mathcal{H}T \tag{3.58}$$

$$\tilde{\chi} = \chi + kL \; . \tag{3.59}$$

From Eqs. (3.56)-(3.59), it is found that the following variables are gauge invariant:

$$\Phi = \phi + \frac{1}{a}\frac{d}{d\tau}\left[ak^{-1}\omega - ak^{-2}\chi'\right] \tag{3.60}$$

$$\Psi = \psi + \frac{1}{3}\chi + \mathcal{H}\left[k^{-1}\omega - k^{-2}\chi'\right] \; . \tag{3.61}$$

It is easy to see Eqs. (3.60) and (3.61) are gauge invariant. Φ and Ψ transform as

$$\begin{aligned}\tilde{\Phi} &= (\phi - T' - \mathcal{H}T) + \left[ak^{-1}(\omega + L' + kT) - ak^{-2}(\chi' + kL')\right] \\ &= \phi - T' - \mathcal{H}T + \left[ak^{-1}\omega + ak^{-1}L' + aT - ak^{-2}\chi' - ak^{-1}L')\right] \\ &= \phi + \frac{1}{a}\frac{d}{d\tau}\left[ak^{-1}\omega - ak^{-2}\chi'\right] = \Phi \; ,\end{aligned} \tag{3.62}$$

and

$$\begin{aligned}\tilde{\Psi} &= (\psi - \frac{kL}{3} - \mathcal{H}T) + \frac{1}{3}(\chi + kL) + \mathcal{H}\left[k^{-1}(\omega + L' + kT) - k^{-2}(\chi' + kL')\right] \\ &= \psi - \frac{kL}{3} - \mathcal{H}T + \frac{1}{3}\chi + \frac{1}{3}kL + \mathcal{H}\left[k^{-1}\omega + k^{-1}L' + T - k^{-2}\chi' - k^{-1}L'\right] \\ &= \psi + \frac{1}{3}\chi + \mathcal{H}\left[k^{-1}\omega - k^{-2}\chi'\right] = \Psi \; .\end{aligned} \tag{3.63}$$

3.2.2. Energy-momentum tensor

The energy-momentum tensor including perturbations is written as

$$T^{\mu}_{\nu} = (\rho + P)u^{\mu}u_{\nu} - P\delta^{\mu}_{\nu} + \Pi^{\mu}_{\nu} \; , \tag{3.64}$$

where Π^{μ}_{ν} is the anisotropic stress tensor and four velocity u^{μ} is given by

$$u^{\mu} = a^{-1}(1 - \phi Q, vQ^{i}) \; . \tag{3.65}$$

As for u_{μ} we obtain

$$u_{\mu} = g_{\mu\nu}u^{\nu} = a(1 + \phi Q, (\omega - v)Q_{j}) \; . \tag{3.66}$$

Notice that $u^\mu u_\mu = 1$ in the first order. So, the energy-momentum tensor is rewritten as

$$T^0_0 = \rho\,(1+\delta Q) \tag{3.67}$$

$$T^0_j = -(\rho+P)(v-\omega)Q_j \tag{3.68}$$

$$T^i_0 = (\rho+P)vQ^i \tag{3.69}$$

$$T^i_j = -P\left(\delta^i_j + \pi_L Q\delta^i_j + \Pi Q^i_j\right)\ , \tag{3.70}$$

where $\delta = \delta\rho/\rho$ and $\pi_L = \delta P/P$. Here notice that ρ and P are homogeneous energy density and pressure. (Contrarily ρ and P in Eq. (3.64) include inhomogeneous parts $\delta\rho$ and δP.)

Gauge transformation of the energy-momentum tensor is given by

$$\begin{aligned}\tilde{T}^\mu_\nu(\tau,x^i) &= \frac{\partial\tilde{x}^\mu}{\partial x^\alpha}\frac{\partial x^\beta}{\partial\tilde{x}^\nu}T^\alpha_\beta(\tau-TQ,x^i-LQ^i)\\ &\simeq T^\mu_\nu(\tau,x^i)+T^\alpha_\nu(\tau,x^i)\delta x^\mu_{\ ,\alpha}-T^\mu_\alpha(\tau,x^i)\delta x^\alpha_{\ ,\nu}-T^\mu_{\nu,\alpha}\delta x^\alpha\ .\end{aligned} \tag{3.71}$$

Using this transformation law, we obtain

$$\begin{aligned}\tilde{T}^0_0 - T^0_0 &= T^\alpha_0\delta x^0_{\ ,\alpha} - T^0_\alpha\delta x^\alpha_{\ ,0} - T^0_{0,\alpha}\delta x^\alpha\\ &= T^0_0\delta x^0_{\ ,0} - T^0_0\delta x^0_{\ ,0} - T^0_{0,0}\delta x^0\\ &= -\rho' TQ\\ &= 3\mathcal{H}(\rho+P)TQ\ ,\end{aligned} \tag{3.72}$$

and

$$\begin{aligned}\tilde{T}^0_j - T^0_j &= T^\alpha_j\delta x^0_{\ ,\alpha} - T^0_\alpha\delta x^\alpha_{\ ,j} - T^0_{j,\alpha}\delta^\alpha\\ &= T^i_j\delta x^0_{\ ,i} - T^0_0\delta x^0_{\ ,j}\\ &= -P(-kTQ_j) - \rho(-kTQ_j)\\ &= k(\rho+P)TQ_j\ .\end{aligned} \tag{3.73}$$

Here we have used the following equation for $\rho'(\dot\rho)$:

$$\rho' = -3\mathcal{H}(\rho+P)\ , \tag{3.74}$$

$$\dot\rho = -3H(\rho+P)\ , \tag{3.75}$$

which are derived from Eq. (1.51). Therefore, $\delta\rho$, v, δP and Π are transformed as

$$\tilde{\delta} = \delta + 3(1+w)\mathcal{H}T \ , \tag{3.76}$$

$$\tilde{v} = v + L' \ , \tag{3.77}$$

$$\tilde{\pi}_L = \pi_L + 3\frac{c_s^2}{w}(1+w)\mathcal{H}T \ , \tag{3.78}$$

$$\tilde{\Pi} = \Pi \ , \tag{3.79}$$

where we have used the equation of state $w = P/\rho$ and the sound velocity $c_s^2 = \delta P/\delta\rho$.

Obviously Π is gauge invariant. The other gauge invariant variables are given by

$$V = v - k^{-1}\chi' \ , \tag{3.80}$$

$$\Delta = \delta + 3\mathcal{H}(1+w)k^{-1}(v-\omega) \ . \tag{3.81}$$

V in Eq. (3.80) is invariant because

$$\begin{aligned}\tilde{V} &= (v+L') - k^{-1}(\chi' + kL') \\ &= v - k^{-1}\chi' = V \ .\end{aligned} \tag{3.82}$$

V is called gauge-invariant velocity pertubation. Similarly, invariance of Δ in Eq. (3.81) is shown as

$$\begin{aligned}\tilde{\Delta} &= (\delta + 3(1+w)\mathcal{H}T) + 3\mathcal{H}(1+w)k^{-1}[(v+L') - (\omega + L' + kT)] \\ &= \delta + 3\mathcal{H}(1+w)k^{-1}(v-\omega) = \Delta \ .\end{aligned} \tag{3.83}$$

Δ is called gauge-invariant density perturbation.

3.2.3. Newtonian gauge

Since metric perturbations have gauge ambiguities, we need gauge fixing to calculate perturbative quantities. Most convenient choice of gauge is to set $\omega = \chi = 0$. This is called Newtonian gauge. In Newtonian gauge, other metric perturbations are related to gauge-invariant variables as

$$\phi = \Phi \ , \tag{3.84}$$

$$\psi = \Psi \ , \tag{3.85}$$

which leads to

$$ds^2 = a^2[1 + 2\Phi Q]d\tau^2 - a^2[1 + 2\Psi Q](d\vec{x})^2 \ . \tag{3.86}$$

Notice that we only consider a flat universe ($K = 0$). The formula for $K \neq 0$ are found in Appendix A.2. Thus, the metric perturbations in the Newtonian gauge is given by

$$g_{00} = a^2(1 + 2\Phi Q) \ , \tag{3.87}$$

$$g_{0i} = 0 \ , \tag{3.88}$$

$$g_{ij} = -a^2(1 + 2\Psi Q)\delta_{ij} \ . \tag{3.89}$$

The Einstein tensor $G_{\mu\nu}$ is calculated as

$$G^0_0 = \frac{3}{a^2}\left(\frac{a'}{a}\right)^2 - \frac{2}{a^2}\left[3\left(\frac{a'}{a}\right)^2\Phi - 3\frac{a'}{a}\Psi' - k^2\Psi\right]Q, \tag{3.90}$$

$$G^0_i = -\frac{2}{a^2}\left[\frac{a'}{a}k\Phi - k\Psi'\right]Q_i, \tag{3.91}$$

$$G^i_0 = \frac{2}{a^2}\left[\frac{a'}{a}k\Phi - k\Psi'\right]Q^i \ . \tag{3.92}$$

$$\begin{aligned} G^i_j = {} & \frac{1}{a^2}\left[2\frac{a''}{a} - \left(\frac{a'}{a}\right)^2\right]\delta^i_j \\ & - \frac{2}{a^2}\left\{\left[2\frac{a''}{a} - \left(\frac{a'}{a}\right)^2\right]\Phi + \frac{a'}{a}[\Phi' - \Psi'] - \frac{k^2}{3}\Phi - \Psi'' - \frac{a'}{a}\Psi' - \frac{1}{3}k^2\Psi\right\}\delta^i_j Q \\ & + \frac{1}{a^2}k^2(\Phi + \Psi)Q^i_j \ . \end{aligned} \tag{3.93}$$

On the other hand the energy-momentum tensor is written as

$$T^0{}_0 = (1 + \delta Q)\,\rho, \tag{3.94}$$

$$T^0{}_i = -(\rho + p)VQ_i, \tag{3.95}$$

$$T^j{}_0 = (\rho + p)VQ^i, \tag{3.96}$$

$$T^i{}_j = -p\left(\delta^i_j + \pi_L Q\delta^i_j + \Pi Q^i_j\right) \tag{3.97}$$

Einstein equation

The homogeneous part of the Einstein equation $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ leads to

$$\mathcal{H}^2 = \frac{8\pi G}{3}a^2\rho \ , \tag{3.98}$$

$$\mathcal{H}' = -\frac{4\pi G}{3}a^2(\rho + 3P) \ , \tag{3.99}$$

where we have used $\mathcal{H} = a'/a$ and $\mathcal{H}' + \mathcal{H}^2 = a''/a$. The above equations correspond to Eqs. (1.47) and (1.49), respectively.

The 1st order contribution from $(0,0)$-component of the Einstein equation is

$$3\mathcal{H}^2\Phi - 3\mathcal{H}\Psi' - k^2\Psi = -4\pi G a^2 \rho\, \delta \; . \tag{3.100}$$

$(0,i)$-component reads

$$\mathcal{H}\Phi - \Phi' = 4\pi G a^2 (1+w) k^{-1} \rho V \; . \tag{3.101}$$

The traceless part of (i,j)-component reads

$$k^2(\Phi + \Psi) = -8\pi G a^2 P \Pi \; . \tag{3.102}$$

Thus, for a vanishing anisotropic tensor we have

$$\Psi = -\Phi \; . \tag{3.103}$$

From Eqs. (3.100) and (3.101) the following Poisson equation is derived:

$$k^2\Psi = 4\pi G a^2 \rho[\delta + 3\mathcal{H}(1+w)k^{-1}V] = 4\pi G a^2 \rho \Delta \; . \tag{3.104}$$

Scalar field

For a scalar field ϕ (do not confuse with the metric perturbation ϕ), the energy-momentum tensor is written as

$$T^{\mu}_{\ \nu} = g^{\mu\lambda}\phi_{,\lambda}\phi_{,\nu} - \delta^{\mu}_{\ \nu}\left(\frac{1}{2}g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - V(\phi)\right) . \tag{3.105}$$

Let us divide the scalar field into homegeneous and fluctuation parts as $\phi(\tau, \vec{x}) = \bar{\phi}(\tau) + \delta\phi(\tau, \vec{k})\, Q(\vec{k}, \vec{x})$. The $(0,0)$-component of the energy-momentum tensor is then written as

$$\begin{aligned} T^{0}_{\ 0} &= \frac{1}{2}g^{0\alpha}\phi_{,\alpha}\phi_{,0} + \frac{1}{2}g^{ij}\phi_{,i}\phi_{,j} + V \\ &\simeq \frac{1}{2}g^{00}\phi_{,0}\phi_{,0} + V \\ &= \frac{1}{2}a^{-2}(1 - 2\Phi Q)(\bar{\phi}' + \delta\phi' Q)^2 + \bar{V} + V_{\phi}\delta\phi\, Q \; , \end{aligned} \tag{3.106}$$

where $V_\phi = dV/d\phi$. Thus, the 1st order contribution of $T^0_{\ 0}$ is

$$\delta T^{0}_{\ 0} = a^{-2}[-\Phi(\bar{\phi}')^2 + \bar{\phi}'\delta\phi' + a^2 V_\phi \delta\phi]Q \; . \tag{3.107}$$

The $(0,i)$-component reads

$$T^0_{\ i} = g^{0\alpha}\phi_{,\alpha}\phi_{,i} = a^{-2}(1-2\Phi Q)\bar{\phi}'(-k\delta\phi\, Q_i) \ , \tag{3.108}$$

which leads to

$$\delta T^0_{\ i} = -ka^{-2}\bar{\phi}'\delta\phi Q_i \ . \tag{3.109}$$

The (i,j)-component redas

$$\begin{aligned} T^i_{\ j} &= g^{i\alpha}\phi_{,\alpha}\phi,j - \delta^i_{\ j}\left[\frac{1}{2}g^{00}(\phi_{,0})^2 - V\right] \\ &= -\delta^i_{\ j}\left[\frac{1}{2}a^{-2}(1-2\Phi Q)(\bar{\phi}'+\delta\phi' Q)^2 - \bar{V} - V_\phi\delta\phi Q\right] \ . \end{aligned} \tag{3.110}$$

Thus, $\delta T^i_{\ j}$ is given by

$$\delta T^i_{\ j} = \delta^i_{\ j}\, a^{-2}[\Phi(\bar{\phi}')^2 - \bar{\phi}'\delta\phi' + a^2 V_\phi\delta\phi]Q \ . \tag{3.111}$$

From Eq. (3.111), for a scalar field anisotropic stress Π vanishes, i.e.,

$$\Pi = 0 \ . \tag{3.112}$$

From Eqs.(3.107), (3.109) and (3.111) ρ, V and π_L are related to $\delta\phi$ and Φ as

$$\rho\,\delta = a^{-2}[-\Phi(\phi')^2 + \phi'\delta\phi' + a^2 V_\phi\delta\phi] \ , \tag{3.113}$$

$$(1+w)\rho V = ka^{-2}\phi'\delta\phi \ , \tag{3.114}$$

$$\pi_L P = a^{-2}[-\Phi(\phi')^2 + \phi'\delta\phi' - a^2 V_\phi\delta\phi] \ , \tag{3.115}$$

Here and hereafter we rewrite $\bar{\phi}$ as ϕ.

Before considering the 1st order Einstein equation, let us write the zero-th order (homogeneous) Einstein equations. Using the conformal time ρ and P are given by

$$\rho = \frac{1}{2a^2}(\phi')^2 + V \ , \tag{3.116}$$

$$P = \frac{1}{2a^2}(\phi')^2 - V \ . \tag{3.117}$$

From Eqs. (3.98) and (3.99) the homogeneous Einstein equations are written as

$$\mathcal{H}^2 = \frac{8\pi G}{3}\left(\frac{1}{2}(\phi')^2 + a^2 V\right) \ , \tag{3.118}$$

$$\mathcal{H}' = \frac{8\pi G}{3}\left(-(\phi')^2 + a^2 V\right) \ , \tag{3.119}$$

from which we obtain

$$\mathcal{H}^2 - \mathcal{H}' = 4\pi G(\phi')^2 \ . \tag{3.120}$$

As for the 1st order Einstein equation, from $(0,0)$-component we obtain

$$3\mathcal{H}^2\Phi - 3\mathcal{H}\Psi' - k^2\Psi = -4\pi G[-\Phi(\phi')^2 + \phi'\delta\phi' + a^2V_\phi\delta\phi] \ . \tag{3.121}$$

From $(0,i)$-component,

$$\mathcal{H}\Phi - \Psi' = 4\pi G\phi'\delta\phi \ . \tag{3.122}$$

From the traceless part of (i,j) component

$$\Psi + \Phi = 0 \ , \tag{3.123}$$

which leads to

$$\boxed{\Phi = -\Psi} \ . \tag{3.124}$$

Using Eqs. (3.120) and (3.124), Eq. (3.121) is rewritten as

$$(\mathcal{H}' + 2\mathcal{H}^2)\Psi + 3\mathcal{H}\Psi' + k^2\Psi = 4\pi G(\phi'\delta\phi' + a^2V_\phi\delta\phi) \ . \tag{3.125}$$

Eq. (3.122) is written as

$$\boxed{\Psi' + \mathcal{H}\Psi = -4\pi G\phi'\delta\phi} \ . \tag{3.126}$$

By differentiating this equation with respect to τ we obtain

$$\begin{aligned}\Psi'' + \mathcal{H}\Psi' + \mathcal{H}'\Psi &= -4\pi G\phi''\delta\phi - 4\pi G\phi'\delta\phi' \\ &= 4\pi G(2\mathcal{H}\phi' + a^2V_\phi)\delta\phi - 4\pi G\phi'\delta\phi' \ ,\end{aligned} \tag{3.127}$$

where in the last line we have used the equation of motion for ϕ,

$$\phi'' + 2\mathcal{H}\phi' + a^2V_\phi = 0 \ . \tag{3.128}$$

Using Eqs. (3.125) and (3.126), Eq. (3.127) is written as

$$\begin{aligned}\Psi'' + \mathcal{H}\Psi' + \mathcal{H}'\Psi = &- 2\mathcal{H}(\Psi' + \mathcal{H}\Psi) \\ &- [(\mathcal{H}' + 2\mathcal{H}^2)\Psi + 3\mathcal{H}\Psi' + k^2\Psi] \\ &+ 8\pi Ga^2V_\phi\delta\phi \ .\end{aligned} \tag{3.129}$$

Moreover using

$$\begin{aligned}8\pi Ga^2V_\phi\delta\phi &= \frac{2(\Psi' + \mathcal{H}\Psi)}{\phi'}(\phi'' + 2\mathcal{H}\phi') \\ &= 4\mathcal{H}\Psi' + 4\mathcal{H}^2\Psi + (2\Psi' + 2\mathcal{H}\Psi)\frac{\phi''}{\phi'} \ ,\end{aligned} \tag{3.130}$$

we obtain

$$\boxed{\Psi'' + \left(2\mathcal{H} - 2\frac{\phi''}{\phi'}\right)\Psi' + \left(2\mathcal{H}' - 2\mathcal{H}\frac{\phi''}{\phi'}\right)\Psi + k^2\Psi = 0} \,. \tag{3.131}$$

In the long wavelength limit $k \to 0 (k \ll \mathcal{H})$ Eq (3.131) has the following solution:

$$\Psi = A\left(1 - \frac{\mathcal{H}}{a^2}\int a^2 d\tau\right) = A\left(1 - \frac{\dot{a}}{a^2}\int a dt\right) \,, \tag{3.132}$$

where $\dot{} = d/dt$. The integration constant A is determined using Eq. (3.126), whihc leads to

$$A = \frac{1}{a^{-2}\int a^2 d\tau}\frac{\delta\phi}{\phi'} = \frac{1}{a^{-1}\int a dt}\frac{\delta\phi}{\dot{\phi}} \simeq H\frac{\delta\phi}{\dot{\phi}} \,, \tag{3.133}$$

where we have used $a^{-1}\int a dt = a^{-1}\int a(dt/da)da = a^{-1}\int H^{-1}da \simeq H^{-1}$. Usually A is estimated when the physical wavelength becomes equal to the Hubble radius $(k/a = H)$ during inflation,

$$A \simeq H\frac{\delta\phi}{\dot{\phi}}\bigg|_{k/a=H} \,. \tag{3.134}$$

Moreover, Eq. (3.132) does not depend on ϕ, so it applies to epochs after inflation. In particular, after reheating the scale factor evolves as $a^m (m = 1/2(\mathrm{RD})$ or $m = 2/3(\mathrm{MD})$ and hence

$$\Psi = A\left(1 - \frac{mt^{m-1}}{t^{2m}}\int t^m dt\right) = \frac{A}{m+1} \,. \tag{3.135}$$

Therefore, we obtain

$$\boxed{\Psi = -\Phi = \begin{cases} \frac{2}{3}H\frac{\delta\phi}{\dot{\phi}}\Big|_{k/a=H} & (\mathrm{RD}) \\ \frac{3}{5}H\frac{\delta\phi}{\dot{\phi}}\Big|_{k/a=H} & (\mathrm{MD}) \end{cases}} \,. \tag{3.136}$$

3.3. ΔN Formula

3.3.1. Basic formulation

Wavelengths of the metric perturbations we are interested in become much longer than the horizon (= Hubble radius) during inflation. The ΔN formula provides a very simple method to calculate such long-wave modes of the fluctuations.

Let us consider the following $(3+1)$-decomposition of the metric:

$$\begin{aligned} ds^2 &= \mathcal{N}^2 dt^2 - \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \\ &= (\mathcal{N}^2 - \beta^k\beta_k)dt^2 - 2\beta_j dt dx^j - \gamma_{ij}dx^i dx^j \,, \end{aligned} \tag{3.137}$$

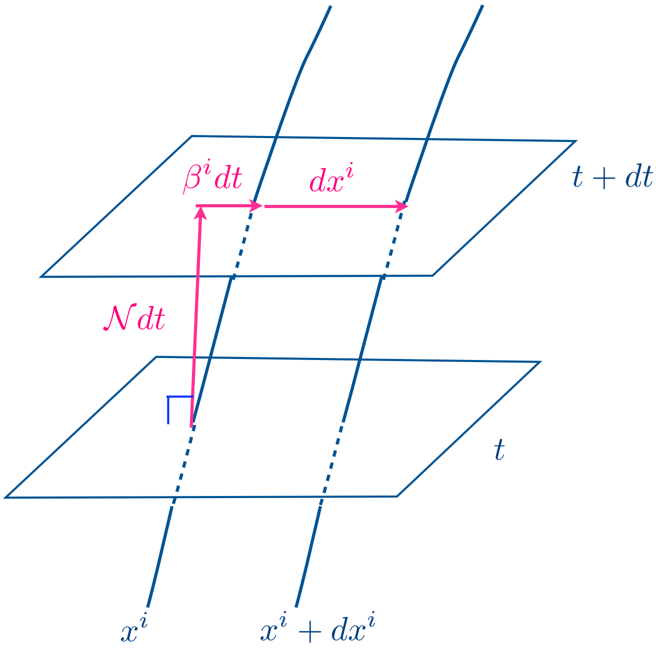

Figure 3.1.: geometrical meaning of the metric (3.137).

where $\mathcal{N}$ is the lapse function, β^i is the shift vector, γ_{ij} is the spatial three metric and $\beta_i = \gamma_{ij}\beta^j$. Their geometrical meaning is shown in Fig. 3.1. Moreover, γ_{ij} is written as

$$\gamma_{ij} = \tilde{a}(t, x^j)\tilde{\gamma}_{ij}(t, x^j) \ , \tag{3.138}$$

where $\det\tilde{\gamma}_{ij} = 1$ and $\tilde{a}$ is the local scale factor which is further written as

$$\tilde{a}(t, x^j) = a(t) \exp[\psi(t, x^j)] \ . \tag{3.139}$$

Here $a(t)$ is the usual scale factor (global scale factor) and ψ is the curvature perturbation. As for $\tilde{\gamma}_{ij}$ it is written as

$$\tilde{\gamma} = I e^{\chi} \qquad\qquad (I : \text{unit matrix}, \quad \text{tr}(\chi) = 0) \ , \tag{3.140}$$

where $\text{tr}(\chi) = 0$ comes from $\det[e^{\chi}] = e^{\text{tr}(\chi)} = 1$. Notice that for small ψ and χ, γ_{ij} is given by

$$\gamma_{ij} = a^2(t)[(1 + 2\psi)\delta_{ij} + \chi_{ij}] \ . \tag{3.141}$$

When one considers the metric perturbations with super-horizon scales ($k/a \ll H$), local homogeneity and isotropy are a good approximation. Thus locally measurable parts of the universe is described by the Robertson-Walker metric

$$ds^2 = dt^2 - a^2(t)\delta_{ij}dx^i dx^j \ . \tag{3.142}$$

This implies that β_i in Eq. (3.137) vanishes in the limit of $k/(aH) \to 0$, which leads to

$$\beta_i = O(\epsilon) \qquad \text{where} \quad \epsilon \equiv \frac{k}{aH} \ . \tag{3.143}$$

Furthermore, we can take the spatial coordinates that comove with the cosmic fluid, which means that the velocity v^i satisfies

$$v^i = \frac{dx^i}{dt} = 0 \ . \tag{3.144}$$

So the spatial components of the four-velocity $u^\mu = dx^\mu/d\tau_p$ (τ_p: proper time$= \int \sqrt{\mathcal{N}^2 - \beta^k\beta_k}dt$) vanish and using $g_{\mu\nu}u^\mu u^\nu = 1$ we obtain

$$u^\mu = \left[\frac{1}{\sqrt{\mathcal{N}^2 - \beta^k\beta_k}},\ 0\right] = \left[\frac{1}{\mathcal{N}},\ 0\right] + O(\epsilon^2) \ . \tag{3.145}$$

The expansion θ is defined as

$$\begin{aligned}\theta \equiv \nabla_\mu u^\mu &= \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}u^\mu) \\ &= \frac{1}{\mathcal{N}e^{2\psi}a^3}\partial_0\left(\frac{\mathcal{N}e^{3\psi}a^3}{\sqrt{\mathcal{N}^2 - \beta^k\beta_k}}\right) \ .\end{aligned} \tag{3.146}$$

Therefore, the expansion is given by

$$\theta = \frac{1}{\mathcal{N}}\left(3\frac{\dot{a}}{a} + 3\dot{\psi}\right) + O(\epsilon^2) \ . \tag{3.147}$$

From this equation we can call $\tilde{H} = \theta/3$ as local Hubble parameter.

The energy-momentum tensor is given by

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu - g_{\mu\nu}P \ , \tag{3.148}$$

where we neglect the anisotropic stress tensor. The energy conservation is given by $\nabla_\nu T^{\mu\nu} = 0$ which leads to

$$\begin{aligned}u_\mu\nabla_\nu T^{\mu\nu} =& 0 \\ =& u_\mu\nabla_\nu[(\rho + P)u^\mu u^\nu - Pg^{\mu\nu}] \\ =& u^\nu u_\mu u^\mu \nabla_\nu(\rho + P) + u_\mu(\rho + P)u^\mu\nabla_\nu u^\nu \\ &+ (u_\mu\nabla_\nu u^\mu)u^\nu(\rho + P) - u_\mu g^{\mu\nu}\nabla_\nu P \ .\end{aligned} \tag{3.149}$$

Using $u^\nu\nabla_\nu = (dx^\nu/d\tau_p)\partial/\partial x^\nu = d/d\tau_p$ and $u_\mu\nabla_\nu u^\mu = \nabla_\nu(u_\mu u^\mu/2) = 0$, the above equation is written as

$$\frac{d}{d\tau_p}(\rho + P) + (\rho + P)\theta - \frac{dP}{d\tau_p} = 0 \ . \tag{3.150}$$

Using Eqs. (3.147) and (3.150) and taking into account $d\tau_p = \sqrt{\mathcal{N}^2 - \beta^k\beta_k}dt \simeq \mathcal{N}dt$, we obtaion

$$\boxed{\frac{\dot{a}}{a} + \dot{\psi} = -\frac{1}{3}\frac{\dot{\rho}}{\rho + P} + O(\epsilon^2) = \frac{\mathcal{N}\theta}{3} + O(\epsilon^2)} \ . \tag{3.151}$$

Let us calculate the local e-folding number defined as

$$N(t_f, t_i, \vec{x}) = \frac{1}{3}\int_{t_i}^{t_f} \theta\mathcal{N}dt \ . \tag{3.152}$$

Using Eq. (3.151) $N(t_f, t_i, \vec{x})$ is calculated as

$$\begin{aligned} N(t_f, t_i, \vec{x}) &= -\frac{1}{3}\int_{t_i}^{t_f} \frac{\dot{\rho}}{\rho + P}dt \\ &= \int_{t_i}^{t_f}\left(\frac{\dot{a}}{a} + \dot{\psi}\right)dt \\ &= \ln\left[\frac{a(t_f)}{a(t_i)}\right] + \psi(t_f, \vec{x}) - \psi(t_i, \vec{x}) \ . \end{aligned} \tag{3.153}$$

Defining N_0 and ΔN as $N_0(t_f, t_i) = \ln[a(t_f)/a(t_i)]$ and $\Delta N(t_f, t_i, \vec{x}) = N(t_f, t_i, \vec{x}) - N_0(t_f, t_i)$,

$$\psi(t_f, \vec{x}) - \psi(t_i, \vec{x}) = \Delta N(t_f, t_i, \vec{x}) \ . \tag{3.154}$$

Now we introduce ζ which is the curvature perturbation on the uniform density slice as

$$\boxed{\zeta \equiv \psi|_{\delta\rho=0}} \ . \tag{3.155}$$

If we take the flat slice ($\psi = 0$) at $t = t_i$ and the uniform density slice ($\delta\rho = 0$) at $t = t_f$, we have

$$\zeta = \Delta N(t_f, t_i, \vec{x}) \ . \tag{3.156}$$

Let us consider the case where the pressure is a function of the energy density, i.e. $P = P(\rho)$. This is satisfied when the perturbations are adiabatic. For generic slicing, Eq. (3.153) is rewritten as

$$\psi(t_f, \vec{x}) - \psi(t_i, \vec{x}) = -\frac{1}{3}\int_{\rho(t_i,\vec{x})}^{\rho(t_f,\vec{x})} \frac{d\rho}{\rho + P} + \frac{1}{3}\int_{\rho(t_i)}^{\rho(t_f)} \frac{d\rho}{\rho + P} \ , \tag{3.157}$$

where we have used $\dot{\rho} = -3(\dot{a}/a)(\rho + P)$. This implies that $\zeta(\vec{x})$ given by

$$\boxed{\zeta(\vec{x}) = \psi(t, \vec{x}) + \frac{\delta\rho}{3(\rho + P)}} \tag{3.158}$$

is conserved independently of choice of time slicing. ζ in Eq. (3.155) is consistent with that in Eq. (3.158). Notice that we have only used the energy conservation and long wave limit. Therefor the result applies to generic gravitation theories including Einstein gravity.

3.3.2. Relation to the Newtonian gauge

In the Newtonian gauge the metric is give by

$$ds^2 = a^2[1 + 2\Phi]d\tau^2 - a^2\delta_{ij}[1 + 2\Psi]dx^i dx^j \ . \tag{3.159}$$

On the other hand the ΔN formula the metric is

$$ds^2 = \cdots \quad - a^2 e^{2\psi}\gamma_{ij}dx^i dx^j = \cdots \quad - a^2[(1 + 2\psi)\delta_{ij} + \chi_{ij}] \ . \tag{3.160}$$

Thus, ζ in Newtonian gauge is given by

$$\zeta = \Psi + \frac{\delta\rho}{3(\rho + P)} \ . \tag{3.161}$$

From the Einstein equation in the Newtonian gauge [Eq. (3.100)]

$$3\mathcal{H}\Phi - 3\mathcal{H}\Psi' - k^2\Psi = -4\pi G a^2 \delta\rho \ . \tag{3.162}$$

For a vanishing anisotropic stress tensor, we get

$$\Psi = -\Phi \ . \tag{3.163}$$

On super-horizon scale ($k \ll H/a$), assuming $\Psi' = 0$, we obtain

$$\Phi = -\frac{4\pi G a^2}{3\mathcal{H}^2}\delta\rho \ . \tag{3.164}$$

Using the Friedmann equation (3.98), Φ is written as

$$\Phi = -\frac{1}{2}\frac{\delta\rho}{\rho} = -\Psi \ . \tag{3.165}$$

From Eq. (3.161)

$$\zeta = \Psi + \frac{\delta\rho}{3(\rho + P)} = -\Phi - \frac{2\Phi}{3(1 + w)} \ . \tag{3.166}$$

So ζ and Φ are related by

$$\boxed{\zeta = -\frac{5 + 3w}{3(1 + w)}\Phi = \frac{5 + 3w}{3(1 + w)}\Psi} \ . \tag{3.167}$$

Since ζ is conserved, the assumption $\Psi' = 0$ is justified for a constant w.

3.3.3. Curvature perturbations produced by inflation

Let us estimate the curvature perturbations produced by the inflaton fluctuations. We take the flat slice at t_i when the relevant scale becomes super-horizon ($k/a \simeq H$) and the uniform density slice at t_f after reheating. From Eqs. (3.156) and (2.110) the curvature perturbation at t_f is given by

$$\zeta = \Delta N = \frac{V}{V' M_G}\delta\phi = -\frac{H_{\rm inf}}{\dot{\phi}}\delta\phi \ . \tag{3.168}$$

Using Eq. (3.167) we obtain

$$\boxed{\Phi = \begin{cases} \frac{2}{3} H_{\rm inf} \left.\frac{\delta\phi}{\dot{\phi}}\right|_{k=aH} & \text{(RD)} \\ \frac{3}{5} H_{\rm inf} \left.\frac{\delta\phi}{\dot{\phi}}\right|_{k=aH} & \text{(MD)} \end{cases}} \ . \tag{3.169}$$

Once Φ is obtained, the density perturbation is calculated using the Poisson equation (3.104),

$$\begin{aligned} k^2\Phi &= -4\pi G a^2 \rho \left[\delta + 3\mathcal{H}(1+w)\frac{V}{k}\right] \\ &= -4\pi G a^2 \rho \Delta \ , \end{aligned} \tag{3.170}$$

where Δ is the gauge invariant density perturbation. Using the Friedmann equation the density perturbation Δ in the matter dominated era is written as

$$\Delta = -\frac{2}{3}\frac{k^2}{a^2H^2}\Phi = -\frac{2}{5}\frac{k^2}{a^2H^2}H_{\rm inf}\frac{\delta\phi}{\dot{\phi}} \qquad \text{(MD)} \ . \tag{3.171}$$

The power spectrum of the density perturbations $\mathcal{P}(k)$ is then given by

$$\langle \Delta(\vec{k})\, \Delta(\vec{k}')\rangle = (2\pi)^3 \delta(\vec{k}+\vec{k}')\frac{2\pi^2}{k^3}\mathcal{P}(k) \ . \tag{3.172}$$

Using Eqs(3.25), (3.27) and (3.171) the power spectrum is written as

$$\mathcal{P}(k) = \frac{4}{25}\left(\frac{k^2}{a^2H^2}\right)^2 \frac{H_{\rm inf}^2}{(\dot{\phi})^2}\mathcal{P}_{\delta\phi}(k) \ . \tag{3.173}$$

Thus, we obtain

$$\boxed{\mathcal{P}(k) = \frac{1}{25\pi^2}\left(\frac{k^2}{a^2H^2}\right)^2 \frac{H_{\rm inf}^4}{(\dot{\phi})^2} = \frac{1}{25\pi^2}\left(\frac{k^2}{a^2H^2}\right)^2 \frac{V^3}{3(V')^2 M_G^6}} \ . \tag{3.174}$$

Similarly the power spectrum of the curvature perturbations $\mathcal{P}_\zeta$ is defined as

$$\langle\zeta(\vec{k})\,\zeta(\vec{k}')\rangle = (2\pi)^3\delta(\vec{k}+\vec{k}')\frac{2\pi^2}{k^3}\mathcal{P}_\zeta(k)\ . \tag{3.175}$$

From Eqs(3.25), (3.27) and (3.168) we obtain

$$\boxed{\mathcal{P}_\zeta(k) = \frac{H_{\rm inf}^4}{4\pi^2(\dot{\phi})^2} = \frac{V^3}{12\pi^2(V')^2M_G^6}}\ . \tag{3.176}$$

In Eqs. (3.174) and (3.176) the values of the inflaton potential V and its derivative V' are evaluated at $k = aH$. The curvature and density perturbations produce anisotropies of the cosmic microwave background (CMB). The observation of CMB by the recent Plank satellite provide us the precise value of the amplitude of the curvature perturbations,

$$[\mathcal{P}_\zeta]^{1/2} = 4.93\times 10^{-5}\ , \tag{3.177}$$

at $k_* = 0.002\ \mathrm{Mpc}^{-1}$.

The spectrum index n_s of the curvature perturbations is defined as

$$n_s - 1 = \frac{d\ln\mathcal{P}_\zeta(k)}{d\ln k}\ , \tag{3.178}$$

which means $\mathcal{P}_\zeta \propto k^{n_s-1}$. Using Eq. (3.176) the spectral index is written as

$$n_s - 1 = 2\frac{d\ln(V^{3/2}/V')}{d\ln k}\ . \tag{3.179}$$

The relation between the cosmological scale $L = k^{-1}$ and the efold N is given by

$$N \sim 50 + \ln\left(\frac{k^{-1}}{1000\mathrm{Mpc}}\right)\ , \tag{3.180}$$

which leads to

$$d\ln k = -dN = -\frac{V}{V'M_G^2}d\phi\ . \tag{3.181}$$

Thus,

$$\begin{aligned} n_s - 1 &= -2M_G^2\frac{d\ln(V^{3/2}/V')}{(V/V')d\phi} \\ &= -2M_G^2\frac{V'}{V}\left(\frac{3}{2}\frac{V'}{V} - \frac{V''}{V'}\right) \\ &= -3M_G^2\left(\frac{V'}{V}\right)^2 + 2M_G^2\frac{V''}{V}\ . \end{aligned} \tag{3.182}$$

With use of the slow-roll parameters ϵ and η [(2.90 and (2.97)] the spectral index n_s can be expressed as

$$\boxed{n_s = 1 - 6\epsilon + 2\eta}\ . \tag{3.183}$$

3.3.4. Curvature perturbations produced by chaotic inflation

Now let us apply the result in the previous section to chaotic inflation. For a simple chaotic inflation model with potential $V = \frac{1}{2}m^2\phi^2$, using Eq. (3.176) the power spectrum of the curvature perturbations at $k = k_*$ is given by

$$[\mathcal{P}_\zeta]^{1/2} = \frac{\left(\frac{1}{2}m^2\phi_*^2\right)^{3/2}}{2\sqrt{3}\pi m^2\phi_* M_G^3} = \frac{m\phi_*^2}{4\sqrt{6}M_G^3} , \tag{3.184}$$

where ϕ_* is the inflaton field value when the mode with k_* crosses the horizon during inflation ($k_* = aH_{\rm inf}$). The corresponding e-fold N_* is estimated from Eq. (2.133) with $L_* = k_*^{-1} = 500\text{Mpc}$ as

$$N_* \simeq 50 . \tag{3.185}$$

Since N_* is given by

$$N_* \simeq \frac{1}{4M_G^2}\phi_*^2 , \tag{3.186}$$

from which we get $\phi_* \simeq 2M_G\sqrt{N_*} \simeq 14M_G$. Therefore, we estimate the amplitude of the curvature perturbations as

$$[\mathcal{P}_\zeta(k_*)]^{1/2} \simeq \frac{m(14M_G)^2}{4\sqrt{6}\pi M_G^3} . \tag{3.187}$$

Requiring that it agrees with the observed value Eq.(3.177) we can determine the inflaton mass as

$$\boxed{m \simeq 1.9 \times 10^{13}\ \text{GeV}} . \tag{3.188}$$

The slow-roll parameters are given by $\eta = \epsilon = 2M_G/\phi^2$, so using Eq. (3.183) the spectral index is

$$n_s(k_*) = 1 - \frac{8M_G^2}{\phi_*^2} \simeq 0.96 . \tag{3.189}$$

3.4. Tensor perturbations

In this section we consider tensor perturbations (gravitational waves) produced during inflation. The amplitude of the tensor perturbations is directly related to the inflation energy scale as we will see later. Therefor, the detection of the tensor mode is crucial to prove inflation.

3.4.1. Generation of tensor perturbations

Let us consider the Einstein-Hilbert action,

$$S_{\rm E} = \frac{M_G}{2}\int dx^4\sqrt{-g}R\ . \tag{3.190}$$

Introducing tensor metric perturbations h_{ij}, the line element is given by

$$ds^2 = a(\tau)^2[d\tau^2 - (\delta_{ij} + h_{ij})dx^i dx^j]\ . \tag{3.191}$$

h_{ij} satisfies the following traceless and transverse conditions:

$$h^i_i = 0 \qquad \partial^i h_{ij} = 0\ . \tag{3.192}$$

Using the above metric the second order action for h_{ij} is given by

$$S_2 = -\frac{M_G^2}{8}\int d^4x a^2[\partial_\mu h_{ij}\partial^\mu h^{ij}]\ . \tag{3.193}$$

Here the indices of h_{ij} are raised or lowered with flat metric δ_{ij}. From the action (3.193) the equation of motion for h_{ij} is written as

$$h''_{ij} + 2\mathcal{H}h''_{ij} - \nabla^2 h_{ij} = 0\ . \tag{3.194}$$

Let us expand h_{ij} into Fourier modes as

$$h_{ij}(\tau,\vec{x}) = \int\frac{d^3k}{(2\pi)^{3/2}}\left[e^{(+)}_{ij}h^{(+)}(\tau) + e^{(\times)}_{ij}h^{(\times)}(\tau)\right]e^{-i\vec{k}\cdot\vec{x}}\ , \tag{3.195}$$

where $e^{(+)}_{ij}$ and $e^{(\times)}_{ij}$ are polarization tensors corresponding two polarization mode +(plus mode) and $\times$(cross mode). Using two orthogonal unit vectors $\vec{e}^{(1)}$ and $\vec{e}^{(2)}$ that are ortghogonal to $\vec{k}$ (i.e., $\vec{e}^{(1)}\perp\vec{e}^{(2)}, \vec{e}^{(1,2)}\perp\vec{k}$), $e^{(+)}_{ij}$ and $e^{(\times)}_{ij}e$ are written as

$$e^{(+)}_{ij} = \frac{1}{\sqrt{2}}\left[e^{(1)}_i(\vec{k})e^{(1)}_j(\vec{k}) - e^{(2)}_i(\vec{k})e^{(2)}_j(\vec{k})\right] \tag{3.196}$$

$$e^{(\times)}_{ij} = \frac{1}{\sqrt{2}}\left[e^{(1)}_i(\vec{k})e^{(2)}_j(\vec{k}) - e^{(2)}_i(\vec{k})e^{(1)}_j(\vec{k})\right]\ , \tag{3.197}$$

which satisfy

$$e^{(+)}_{ij}e^{(+)}_{ij} = e^{(\times)}_{ij}e^{(\times)}_{ij} = 1, \quad e^{(+)}_{ij}e^{(\times)}_{ij} = 0\ . \tag{3.198}$$

If we take $\vec{k} = (0,0,k)$, $e^{(+)}_{ij}$ and $e^{(\times)}_{ij}e$ are given by

$$e^{(+)}_{ij} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix} \qquad e^{(\times)}_{ij} = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\ . \tag{3.199}$$

The mode functions $h^{(+)}$ and $h^{(\times)}$ satisfy

$$h^{(+,\times)\prime\prime} + 2\mathcal{H}h^{(+,\times)\prime} + k^2 h^{(+,\times)} = 0 \ , \tag{3.200}$$

which is the same equation as that for a massless scalar field, so it has a solution like $h \sim H_{\rm inf}$ at long wave limit. In order to get the correct normalization we have to write the action in the form as

$$S_2 = -\int dx^4 a^2 \left(\frac{1}{2}\partial_\mu \tilde{h}_{ij}\partial^\mu \tilde{h}^{ij}\right) \ , \tag{3.201}$$

where $\tilde{h}_{ij}$ is the canonical field. Comparing with Eq. (3.193)

$$\tilde{h}_{ij} = \frac{M_G}{2}h_{ij} \quad \Rightarrow \quad \tilde{h}^{(+,\times)} = \frac{M_G}{2}h^{(+,\times)} \ . \tag{3.202}$$

Since for the canonical field we get the power spectrum $\mathcal{P}_{\tilde{h}}(k) = H_{\rm inf}^2/(4\pi^2)$, we obtain the power spectra for $h^{(+,\times)}$ as

$$\mathcal{P}_{h^{(+)}}(k) = \mathcal{P}_{h^{(\times)}}(k) = \left(\frac{2}{M_G}\right)^2 \frac{H_{\rm inf}^2}{4\pi^2} = \frac{H_{\rm inf}^2}{\pi^2 M_G^2} = \frac{V}{3\pi^2 M_G^4} \ . \tag{3.203}$$

The total power spectrum of the tensor mode is given by

$$\boxed{\mathcal{P}_h(k) = \mathcal{P}_{h^{(+)}}(k) + \mathcal{P}_{h^{(\times)}}(k) = \frac{2H_{\rm inf}^2}{\pi^2 M_G^2} = \frac{2V}{3\pi^2 M_G^4}} \ . \tag{3.204}$$

Notice that the tensor mode only depends on the Hubble parameter during inflation ($H_{\rm inf} = \sqrt{V/3}/M_G$). Therefore, if the tensor mode is observed we can determine the energy scale of inflation. On the other hand the power spectrum of the curvature perturbations is given by Eq.(3.176) and it depends on V' as well as V. The tensor-to-scalar ration r is

$$\boxed{r \equiv \frac{\mathcal{P}_h}{\mathcal{P}_\zeta} = 8\left(\frac{V'}{V}\right)^2 M_G^2 = 16\epsilon} \ . \tag{3.205}$$

The tensor spectral index $n_{\rm T}$ is given by

$$n_{\rm T} = \frac{d\ln \mathcal{P}_h}{d\ln k} = -M_G^2 \frac{V'}{V}\frac{d\ln V}{d\phi} = -\left(\frac{V'}{V}\right)^2 M_G^2 \ . \tag{3.206}$$

With use of r, $n_{\rm T}$ is written as

$$\boxed{n_{\rm T} = -\frac{r}{8}} \ . \tag{3.207}$$

A. Appendix

A.1. Monopole in $SO(3)$ gauge theory

In this appendix we show the monopole solution in a $SO(3)$ gauge theory.

A.1.1. $SO(3)$ gauge theory

Under $SO(3)$ gauge symmetry, a real scalar field Φ is transffered as

$$\Phi(x) \;\to\; U(x)\Phi(x) \qquad [\Phi^a \;\to\; U_{ab}\Phi^b] \tag{A.1}$$

$$U(x) = \exp[-i\theta^a(x)T^a] \;, \tag{A.2}$$

where g is the gauge coupling constant, θ^a's are the transformation parameters and T^a's are the generators which satisfy

$$[T^a, T^b] = if^{abc}T^c \;, \tag{A.3}$$

where f^{abc}s are the structure constants. In the case of $SO(3)$, we have

$$(T^a)_{bc} = i\epsilon^{abc} \tag{A.4}$$

$$f^{abc} = -\epsilon^{abc} \tag{A.5}$$

$$\mathrm{Tr}[T^aT^b] = 2\delta^{ab} \;. \tag{A.6}$$

The gauge field $A^a_\mu(x)$ is transffred as

$$A_\mu(x) \;\to\; U(x)A_\mu(x)U^\dagger(x) + \frac{i}{g}U(x)\partial_\mu U^\dagger(x) \;, \tag{A.7}$$

where $A_\mu \equiv A^a_\mu T^a$. We introduce the covariant derivative defined as

$$D_\mu = \partial_\mu - igA_\mu \;, \tag{A.8}$$

which leads to

$$(D_\mu\Phi)^a = \partial_\mu\Phi^a - igA^b_\mu(T^b)_{ac}\Phi^c = \partial_\mu\Phi^a - g\epsilon^{abc}A^b_\mu\Phi^c \;. \tag{A.9}$$

Under $SO(3)$, the covariant derivative is transformed as

$$D_\mu \;\rightarrow\; U D_\mu U^{-1} \; . \tag{A.10}$$

The field strength of the gauge field and its transformation are given by

$$F_{\mu\nu} = \frac{i}{g}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \tag{A.11}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\epsilon_{abc} A^b_\mu A^c_\nu \tag{A.12}$$

$$F_{\mu\nu} \;\rightarrow\; U F_{\mu\nu} U^{-1} \; . \tag{A.13}$$

Let us consider the following $SO(3)$ invariant lagrangian:

$$\mathcal{L} = \frac{1}{2}(D^\mu\Phi)^a(D_\mu\Phi)^a - \frac{1}{4}F^a_{\mu\nu}F^{a\;\mu\nu} - V(\Phi) \tag{A.14}$$

$$V(\Phi) = \frac{\lambda}{8}(\Phi^a\Phi^a - \sigma^2)^2 \; , \tag{A.15}$$

where λ is a constant. The potential V has the minimum for

$$|\Phi| = (\Phi^a\Phi^a)^{1/2} = \sigma \; . \tag{A.16}$$

When Φ takes the vacuum expectation value, $SO(3)$ symmetry is spontaneously broken to $SO(2) = U(1)$. This is seen by a concrete example. Suppose Φ takes $(0, 0, \sigma)$, then we still have the following transformation which does not change the vacuum:

$$\Phi'_1 = \cos\beta \;\Phi_1 - \sin\beta \;\Phi_2 \tag{A.17}$$

$$\Phi'_2 = \sin\beta \;\Phi_1 + \cos\beta \;\Phi_2 \; . \tag{A.18}$$

This is the $SO(2)$ transformation. If we define a complex scalar field φ as $\varphi = \Phi_1 + i\Phi_2$, the above transformation is written as

$$\varphi' = e^{i\beta}\varphi \; . \tag{A.19}$$

This is $U(1)$ transformation and shows that $SO(2) = U(1)$.

Here let us see that the gauge bosons get mass after spontaneous symmetry breaking. The kinetic term of the scalar field contains A^2 term as

$$\frac{1}{2}(D_\mu\Phi)^a(D^\mu\Phi)^a \ni \frac{1}{2}g^2\epsilon_{abc}\epsilon_{afg}A^b_\mu A^{f\,\mu}\Phi^c\Phi^g \; . \tag{A.20}$$

When the scalar field takes $\Phi = (0, 0, \sigma)$,

$$
\begin{aligned}
\frac{1}{2}(D_\mu \Phi)^a (D^\mu \Phi)^a \ni & \frac{1}{2} g^2 \sigma^2 \epsilon_{ab3} \epsilon_{af3} A^b_\mu A^{f\,\mu} \\
&= \frac{1}{2} g^2 \sigma^2 (\delta_{bf} - \delta_{b3}\delta_{f3}) A^b_\mu A^{f\,\mu} \\
&= \frac{1}{2} g^2 \sigma^2 (A^1_\mu A^{1\,\mu} + A^2_\mu A^{2\,\mu}) \ . \qquad (A.21)
\end{aligned}
$$

Thus, A^1 and A^2 obtain a mass $g\sigma$ while A^3 remains massless, which implies that A^3 is the gauge field for remaining $U(1)$ symmetry.

A.1.2. Gauge invariant electromagnetic field

Gauge invariant electromagnetic field is written as [3]

$$
\mathcal{F}_{\mu\nu} = \frac{1}{\sigma} \Phi_a F^a_{\mu\nu} + \frac{1}{\sigma^2} \epsilon_{abc} \Phi_a (D_\mu \Phi)_b (D_\nu \Phi)_c \ . \qquad (A.22)
$$

The gauge invariance of $\mathcal{F}_{\mu\nu}$ is is seen as follow. The 2nd term in LHS of Eq. (A.22)) is transformed as

$$
\begin{aligned}
\epsilon_{abc} \Phi_a (D_\mu \Phi)_b (D_\nu \Phi)_c \to & \epsilon_{abc} U_{ai} U_{bj} U_{ck} \Phi_i (D_\mu \Phi)_j (D_\nu \Phi)_k \\
&= \epsilon_{ijk} \Phi_i (D_\mu \Phi)_j (D_\nu \Phi)_k \ , \qquad (A.23)
\end{aligned}
$$

where we have used $\epsilon_{abc} C_{ai} C_{bj} C_{ck} = (\det C) \epsilon_{ijk}$ for a arbitrary matrix C. As for the 1st term,

$$
\begin{aligned}
\Phi^a F^a_{\mu\nu} = \frac{1}{2} \Phi^a \ \mathrm{tr}[F_{\mu\nu} T^a] \to & \frac{1}{2} U_{ai} \Phi^i \ \mathrm{tr}[U F_{\mu\nu} U^\dagger T^a] \\
&= \frac{1}{2} U_{ai} \Phi^i U_{bj} F^d_{\mu\nu} T^d_{jk} U_{ck} T^a_{cb} \\
&= -\frac{1}{2} \Phi^i F^d_{\mu\nu} U_{ai} U_{bj} U_{ck} \epsilon_{djk} \epsilon_{acb} \\
&= \frac{1}{2} \Phi^i F^d_{\mu\nu} \epsilon_{djk} \epsilon_{ijk} = \Phi^i F^i_{\mu\nu} \ . \qquad (A.24)
\end{aligned}
$$

Thus, the both terms are gauge invariant and hence $\mathcal{F}_{\mu\nu}$.

If the scalar field Φ takes vacuum expectation value $\Phi = (0, 0, \sigma)$, $\mathcal{F}_{\mu\nu}$ is calculated as

$$
\begin{aligned}
\mathcal{F}_{\mu\nu} &= F^3_{\mu\nu} + \frac{1}{g\sigma} \epsilon_{3bc} (\partial_\mu \Phi^b - g \epsilon_{b\ell m} A^\ell_\mu \Phi^m)(\partial_\nu \Phi^c - g \epsilon_{cpq} A^p_\nu \Phi^q) \\
&= \partial_\mu A^3_\nu - \partial_\nu A^3_\mu - g \epsilon_{3bc} A^b_\mu AS^c_\nu + g \epsilon_{3bc} \epsilon_{b\ell 3} \epsilon_{cp3} A^\ell_\mu A^p_\nu \\
&= \partial_\mu A^3_\nu - \partial_\nu A^3_\mu - g \epsilon_{3bc} A^b_\mu AS^c_\nu + g \epsilon_{cp3} A^c_\mu A^p_\nu \\
&= \partial_\mu A^3_\nu - \partial_\nu A^3_\mu \ . \qquad (A.25)
\end{aligned}
$$

This is the field strength of electromagnetic field. Notice that with $\Phi = (0, 0, \sigma)$ only A^3 is massless coresponding to $U(1)$ symmetry into which SO(3) is broken.

A.1.3. Monopole solution

In order find a monopole solution let us take the static "Hedgehog" configuration ($A_0 = 0$),

$$\Phi^a(r) \sim \sigma\hat{r}_a = \sigma\frac{x_a}{r} \tag{A.26}$$

$$A_i^a(r) \sim \epsilon_{iab}\frac{\hat{r}_b}{gr} = \epsilon_{iab}\frac{x_b}{gr^2} \ , \tag{A.27}$$

at $r \gg \sigma^{-1}$. With the Hedgehog configuration,

$$\partial_i\Phi^a(r) \sim \partial_i\left(\frac{\sigma x_a}{r}\right) \sim \frac{\sigma}{r}(\delta_{ia} - \hat{r}_a\hat{r}_i) \ . \tag{A.28}$$

and

$$g\epsilon_{abc}A_i^b\Phi^c \sim g\epsilon_{abc}\frac{\epsilon_{ibd}\hat{r}_d}{gr}\sigma\hat{r}_c = \frac{\sigma}{r}(\delta_{ai}\delta_{cd} - \delta_{ad}\delta_{ic})\hat{r}_d\hat{r}_c = \frac{\sigma}{r}(\delta_{ai} - \hat{r}_a\hat{r}_i) \ . \tag{A.29}$$

Thus, at $r \gg \sigma^{-1}$ we obtain

$$(D_i\Phi)^a \sim 0, \qquad V(\Phi) \sim 0 \ , \tag{A.30}$$

which shows that the energy of the scalar field is localized around the center ($r \sim 0$). As for the gauge field, we obtain

$$\partial_i A_j^a \sim \partial_i\left(\epsilon_{jab}\frac{x_b}{gr^2}\right) = \frac{1}{gr^2}(\epsilon_{ija} - 2\epsilon_{jab}\hat{r}_i\hat{r}_b) \ , \tag{A.31}$$

and

$$\begin{aligned} g\epsilon_{abc}A_i^bA_j^c &\sim g\epsilon_{abc}\frac{\epsilon_{ibd}\hat{r}_d}{gr}\frac{\epsilon_{jcf}\hat{r}_f}{gr} \\ &= \frac{1}{gr^2}(\delta_{ai}\delta_{cd} - \delta_{ad}\delta_{ci})\hat{r}_d\hat{r}_f\epsilon_{jcf} \\ &= \frac{1}{gr^2}(\delta_{ai}\hat{r}_c\hat{r}_f\epsilon_{jcf} - \hat{r}_a\epsilon_{jif}\hat{r}_f) \\ &= \frac{1}{gr^2}\epsilon_{ijb}\hat{r}_a\hat{r}_b \ . \end{aligned} \tag{A.32}$$

From Eqs. (A.31) and (A.32)

$$F_{ij}^a \sim \frac{1}{gr^2}(2\epsilon_{ija} - 2\epsilon_{jab}\hat{r}_i\hat{r}_b 2\epsilon_{iab}\hat{r}_j\hat{r}_b - \epsilon_{ijb}\hat{r}_a\hat{r}_b) \ , \tag{A.33}$$

from which we obtain the "magnetic" field,

$$B_k^a = \frac{1}{2}\epsilon_{kij}F_{ij}^a \tag{A.34}$$

$$\sim \frac{1}{gr^2}(2\delta_{ka} - 2(\delta_{ka}\delta_{ib} - \delta_{kb}\delta_{ia})\hat{r}_i\hat{r}_b - \delta_{kb}\hat{r}_a\hat{r}_b) \tag{A.35}$$

$$= \frac{1}{gr^2}\hat{r}_a\hat{r}_k \ . \tag{A.36}$$

From Eq. (A.22) the gauge invariant magnetic field is obtained as

$$\mathcal{B}_k = \frac{1}{\sigma}\Phi^a B_k^a \ \sim \ \frac{1}{gr^2}\hat{r}_k \ . \tag{A.37}$$

This shows that the configuration Eqs. (A.26) and (A.27) has a magnetic charge Q_M given by

$$\boxed{Q_M = \int d^2S\hat{r}_i\mathcal{B}_i = \frac{4\pi}{g}} \ . \tag{A.38}$$

Therefore this is the configuration which represents a magnetic monopole The energy of the monopole is written as

$$E_M = \int \mathcal{H}d^3x = -\int \mathcal{L}d^3x \ , \tag{A.39}$$

where $\mathcal{H}$ is the Hamiltonian density. Taking

$$\xi = g\sigma r \tag{A.40}$$

$$\Phi^a = \sigma H(\xi)\hat{r}^a \tag{A.41}$$

$$A_i^a = \frac{1}{gr}\epsilon_{aij}\hat{r}_j(1 - K(\xi)) \ , \tag{A.42}$$

where $H(\xi)$ and $K(\xi)$ are functions of ξ ($H \to 1$, $K \to 0$ as $r \to \infty$) which are determined by minimizing the energy,

$$E = \frac{4\pi\sigma}{g}\int_0^\infty d\xi \left[(K')^2 + \frac{(K^2-1)^2}{2\xi^2} + H^2K^2 + (\xi H')^2 + \frac{\lambda}{8g^2}\xi^2(H^2-1)^2\right] \ . \tag{A.43}$$

From the above equation the mass of the monopole is estimated as

$$\boxed{M_M \simeq \frac{4\pi\sigma}{g}} \ . \tag{A.44}$$

A.2. Newtonian Gauge

A.2.1. Metric perturbations

Metrics in the Newtonian gauge are given by[1]

$$g_{00} = a^2(1 + 2\Phi Q), \tag{A.45}$$

$$g_{0i} = 0, \tag{A.46}$$

$$g_{ij} = -a^2(1 + 2\Psi Q)\gamma_{ij} \tag{A.47}$$

Correspondingly, $g^{\mu\nu}$ are

$$g^{00} = a^{-2}(1 - 2\Phi Q), \tag{A.48}$$

$$g^{0i} = 0, \tag{A.49}$$

$$g^{ij} = -a^{-2}(1 - 2\Psi Q)\gamma_{ij} \tag{A.50}$$

From them the Christoffel symbols are written as

$$\Gamma^0_{00} = \frac{a'}{a} + \Phi' Q, \tag{A.51}$$

$$\Gamma^0_{0i} = -k\Phi Q_i, \tag{A.52}$$

$$\Gamma^i_{00} = -k\Phi Q^i, \tag{A.53}$$

$$\Gamma^i_{0j} = \left(\frac{a'}{a} + \Psi' Q\right)\delta^i_j, \tag{A.54}$$

$$\Gamma^0_{ij} = \left[\frac{a'}{a} + \left(-2\frac{a'}{a}\Phi + 2\frac{a'}{a}\Psi + \Psi'\right)Q\right]\gamma_{ij}, \tag{A.55}$$

$$\Gamma^i_{jk} =^{(s)} \Gamma^i_{jk} - k\Psi(\delta^i_j Q_k + \delta^i_k Q_j - \gamma_{jk}Q^i) \tag{A.56}$$

Here ${}^{(s)}\Gamma^i_{jk}$ is the Christoffel symbol on the time slice with metric γ_{ij}.

Einstein tensor is written as $G_{\mu\nu} = \bar{G}_{\mu\nu} + \delta G_{\mu\nu}$ where $\bar{G}_{\mu\nu}$ and $\delta G_{\mu\nu}$ are homogeneous and 1st order fluctuation parts, respectively. The homogeneous part is written as

$$\bar{G}^0_0 = \frac{3}{a^2}\left[\left(\frac{a'}{a}\right)^2 + K\right], \tag{A.57}$$

$$\bar{G}^i_j = \frac{1}{a^2}\left[2\frac{a''}{a} - \left(\frac{a'}{a}\right)^2 + K\right]\delta^i_j, \tag{A.58}$$

$$\bar{G}^0_i = \bar{G}^i_0 = 0\ . \tag{A.59}$$

[1]Fomula in this appendix are taken from [4] with some modifications of notation.

The 1st order fluctuations are written as

$$\delta G^0_0 = -\frac{2}{a^2}\left[3\left(\frac{a'}{a}\right)^2\Phi - 3\frac{a'}{a}\Psi' - (k^2-3K)\Psi\right]Q, \tag{A.60}$$

$$\delta G^0_i = -\frac{2}{a^2}\left[\frac{a'}{a}k\Phi - k\Psi'\right]Q_i, \tag{A.61}$$

$$\delta G^i_0 = \frac{2}{a^2}\left[\frac{a'}{a}k\Phi - k\Psi'\right]Q^i, \tag{A.62}$$

$$\begin{aligned}\delta G^i_j = -\frac{2}{a^2}&\left\{\left[2\frac{a''}{a} - \left(\frac{a'}{a}\right)^2\right]\Phi + \frac{a'}{a}[\Phi'-\Psi'] - \frac{k^2}{3}\Phi - \Psi'' - \frac{a'}{a}\Psi'\right.\\ &\left.-\frac{1}{3}(k^2-3K)\Psi\right\}\delta^i_jQ + \frac{1}{a^2}k^2(\Phi+\Psi)Q^i_j,\end{aligned} \tag{A.63}$$

A.2.2. Energy-momentum tensor

The first order perturbations of the energy momentum tensor are given by

$$T^0{}_0 = (1+\delta Q)\,\rho, \tag{A.64}$$

$$T^0{}_i = -(\rho+p)VQ_i, \tag{A.65}$$

$$T^j{}_0 = (\rho+p)VQ^i, \tag{A.66}$$

$$T^i{}_j = -p\left(\delta^i_j + \pi_L Q\delta^i_j + \Pi Q^i_j\right) \tag{A.67}$$

From the energy conservation $T^{\mu 0}{}_{;\mu} = 0$, we obtain

$$\begin{aligned}0 = T^{\mu 0}{}_{;\mu} &= \partial_\mu T^{\mu 0} + \Gamma^0{}_{\alpha\beta}T^{\alpha\beta} + \Gamma^\alpha{}_{\alpha\beta}T^{0\beta}\\ &= (\partial_0 T^{00} + \partial_i T^{i0}) + (\Gamma^0{}_{00}T^{00} + 2\Gamma^0{}_{0i}T^{0i} + \Gamma^0{}_{ij}T^{ij})\\ &+ (\Gamma^0{}_{00}T^{00} + \Gamma^0{}_{0i}T^{0i} + \Gamma^i{}_{i0}T^{00} + \Gamma^i{}_{ij}T^{0j})\\ &= \partial_0 T^{00} + \partial_i T^{i0} +^{(s)} \Gamma^i{}_{ij}T^{0j} + 2\Gamma^0{}_{00}T^{00} + \Gamma^0{}_{ij}T^{ij} + \Gamma^i{}_{i0}T^{00}\\ &= \partial_0 T^{00} + T^{i0}{}_{|i} + 2\Gamma^0{}_{00}T^{00} + \Gamma^0{}_{ij}T^{ij} + \Gamma^i{}_{i0}T^{00}\end{aligned} \tag{A.68}$$

where

$$T^{00} = a^{-2}(1+\delta Q - 2\Phi Q)\rho, \tag{A.69}$$

$$\partial_0 T^{00} = a^{-2}\left[(1+\delta Q - 2\Phi Q)\left(\frac{\rho'}{\rho} - 2\frac{a'}{a}\right) + (\delta' - 2\Phi' Q)\right]\rho, \tag{A.70}$$

$$T^{00}_{\ \ |i} = a^{-2}(1+w)kVQ\rho, \tag{A.71}$$

$$\Gamma^0{}_{00}T^{00} = a^{-2}\left[\frac{a'}{a}(1+\delta Q - 2\Phi Q) + \Phi' Q\right]\rho, \tag{A.72}$$

$$\Gamma^0{}_{ij}T^{ij} = 3a^{-2}w\left[\frac{a'}{a}(1+\pi_L Q - 2\Phi Q) + \Psi' Q\right]\rho, \tag{A.73}$$

$$\Gamma^i{}_{i0}T^{00} = 3a^{-2}\left[\frac{a'}{a}(1+\delta Q - 2\Phi Q) + \Psi' Q\right]\rho\ , \tag{A.74}$$

from which 0-th and 1st order equations are onbtained as

$$\rho' = -3(1+w)\frac{a'}{a}\rho \tag{A.75}$$

$$\delta' = -(1+w)(kV + 3\Psi') - 3\frac{a'}{a}\delta w\ . \tag{A.76}$$

Here δw is the fluctuation in the equation of state.

The momentum conservation $T^{\mu i}{}_{;\mu} = 0$ is written as

$$\begin{aligned}
0 = T^{\mu i}{}_{;\mu} &= \partial_\mu T^{\mu i} + \Gamma^i{}_{\alpha\beta}T^{\alpha\beta} + \Gamma^\alpha{}_{\alpha\beta}T^{i\beta} \\
&= (\partial_0 T^{0i} + \partial_j T^{ji}) + (\Gamma^i{}_{00}T^{00} + 2\Gamma^i{}_{0j}T^{0j} + \Gamma^i{}_{jk}T^{jk}) \\
&+ (\Gamma^0{}_{00}T^{0i} + \Gamma^0{}_{0j}T^{ji} + \Gamma^j{}_{j0}T^{0i} + \Gamma^k{}_{kj}T^{ij}) \\
&= \partial_0 T^{0i} + T^{ji}{}_{|j} + \Gamma^i{}_{00}T^{00} + 2\Gamma^i{}_{0j}T^{0j} + \Gamma^0{}_{00}T^{0i} + \Gamma^0{}_{0j}T^{ji} + \Gamma^j{}_{j0}T^{0i}\ ,
\end{aligned} \tag{A.77}$$

where

$$\partial_0 T^{0i} = a^{-2}\left\{\left[(1+w)\left(\frac{\rho'}{\rho} - 2\frac{a'}{a}\right) + w'\right]VQ^i + (1+w)V'Q^i\right\}\rho \tag{A.78}$$

$$T^{ji}{}_{|j} = a^{-2}\left[-\pi_L + \frac{2}{3}(1-3K/k^2)\Pi + 2\Psi\right]wkQ^i\rho \tag{A.79}$$

$$\Gamma^i{}_{00}T^{00} = -a^{-2}k\Phi Q^i\rho \tag{A.80}$$

$$\Gamma^i{}_{0j}T^{0j} = \Gamma^0{}_{00}T^{0i} = \frac{1}{3}\Gamma^j{}_{j0}T^{0i} = a^{-2}\left(\frac{a'}{a}\right)(1+w)VQ^i\rho\ . \tag{A.81}$$

This leads to

$$V' = -\frac{a'}{a}(1-3w)V - \frac{w'}{1+w}V + \frac{\delta p/\delta\rho}{1+w}k\delta - \frac{2}{3}\frac{w}{1+w}(1-3K/k^2)k\Pi + k\Phi \tag{A.82}$$

A.2.3. Einstein equation

Let us write the 1st order Einstein equation. From $(0,0)$ component,

$$3\left(\frac{a'}{a}\right)^2\Phi - 3\frac{a'}{a}\Psi' - (k^2 - 3K)\Psi = -4\pi G a^2 \rho\delta \tag{A.83}$$

From $(0,i)$ component,

$$\frac{a'}{a}\Phi - \Psi' = 4\pi G a^2 (1+w)\rho V/k \tag{A.84}$$

From traceless (i,j) component,

$$k^2(\Phi + \Psi) = -8\pi G a^2 p\Pi \tag{A.85}$$

From Eqs (A.83) and (A.84) we obtain Poisson equation,

$$(k^2 - 3K)\Psi = 4\pi G a^2 \rho\left[\delta + 3\frac{a'}{a}(1+w)V/k\right] \tag{A.86}$$

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