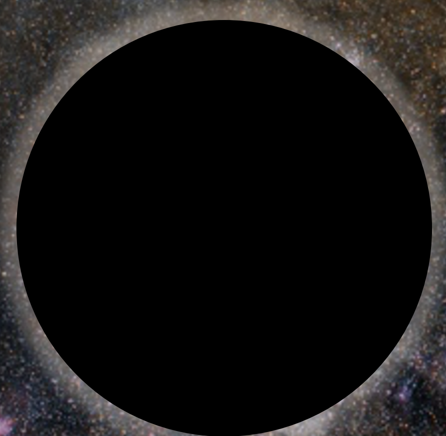
Exact solution for wave scattering from black holes



Hayato Motohashi

(Kogakuin Univ.)

HM, Sousuke Noda, arXiv:2101.xxxxx

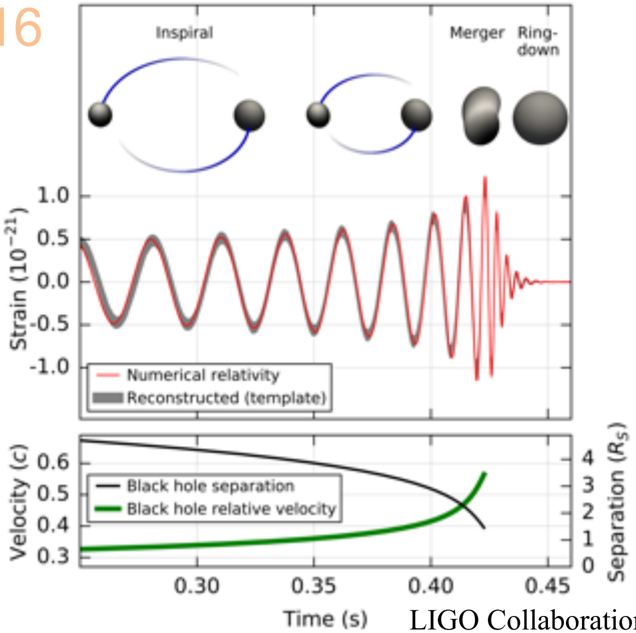
2021.1.18 ICRR workshop "Black Hole Astrophysics with VLBI: Multi-Wavelength and Multi-Messenger Era"

KOGAKUIN UNIVERSITY

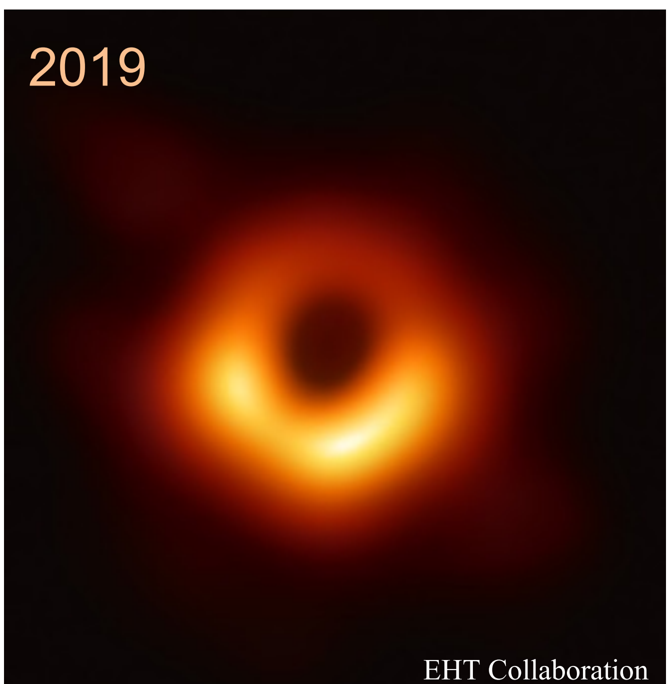
科研費 KAKENHI

Contents

2016



LIGO Collaboration



The Nobel Prize in Physics 2017

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Rainer Weiss

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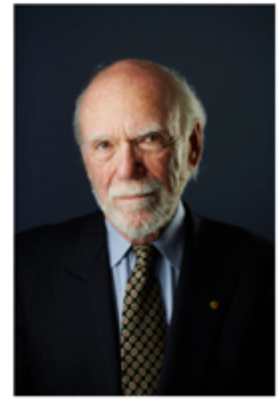

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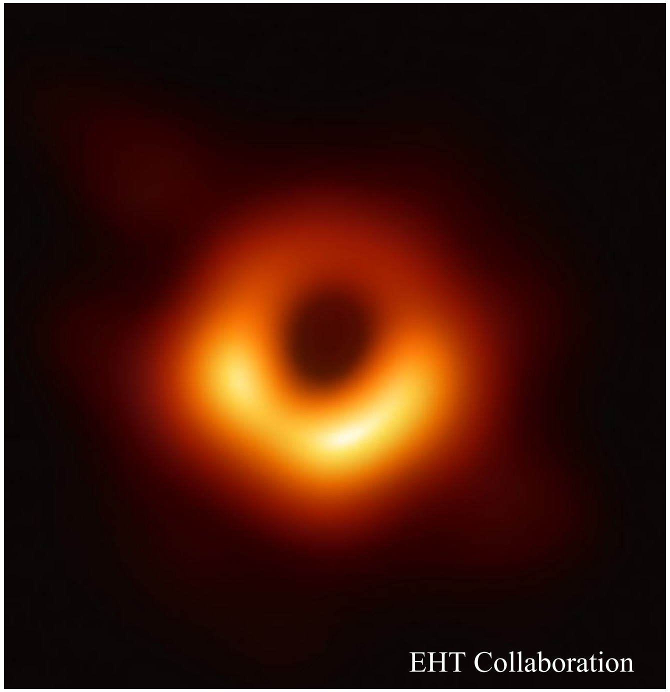
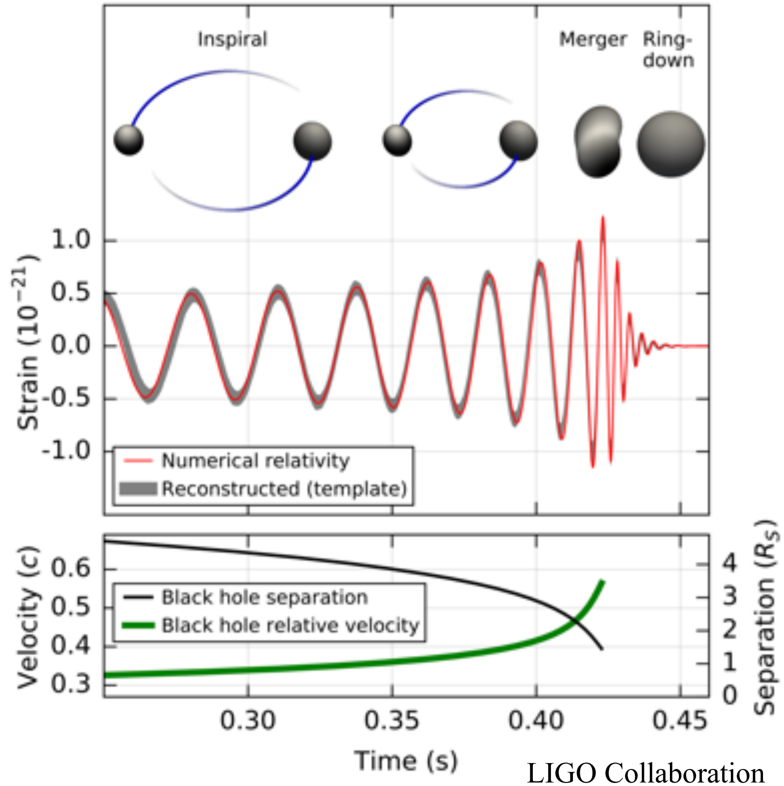


Background spacetime geometry



Perturbations

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

General Relativity

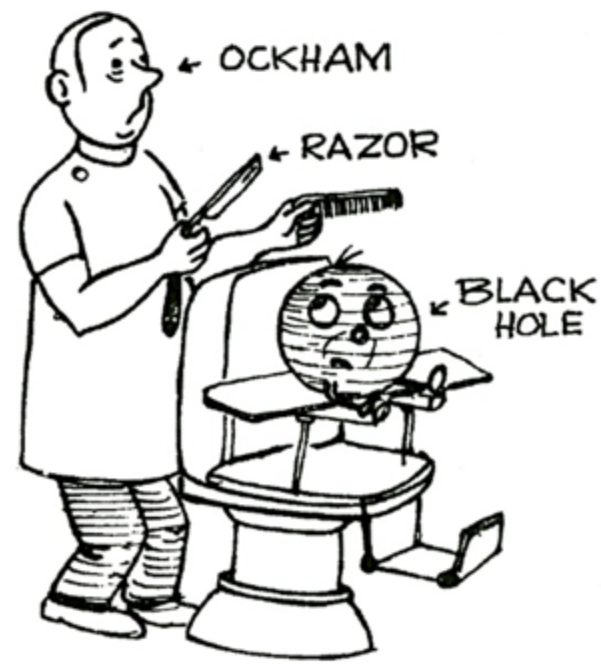
The uniqueness theorem

Stationary axisym. BH in GR

$$\Downarrow$$

$\exists !$ $\boxed{\text{Kerr solution}}$

M : Mass
a : Angular momentum

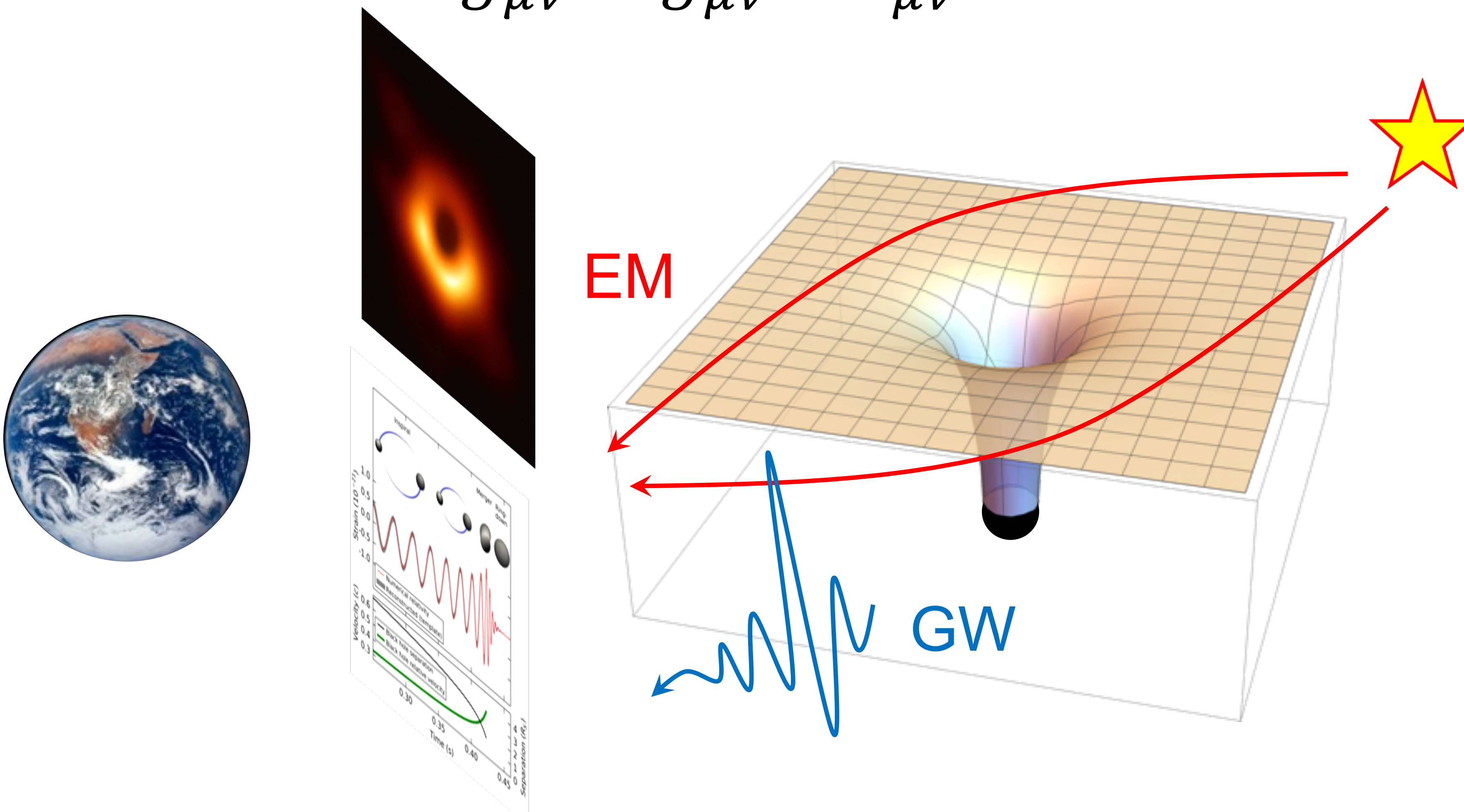


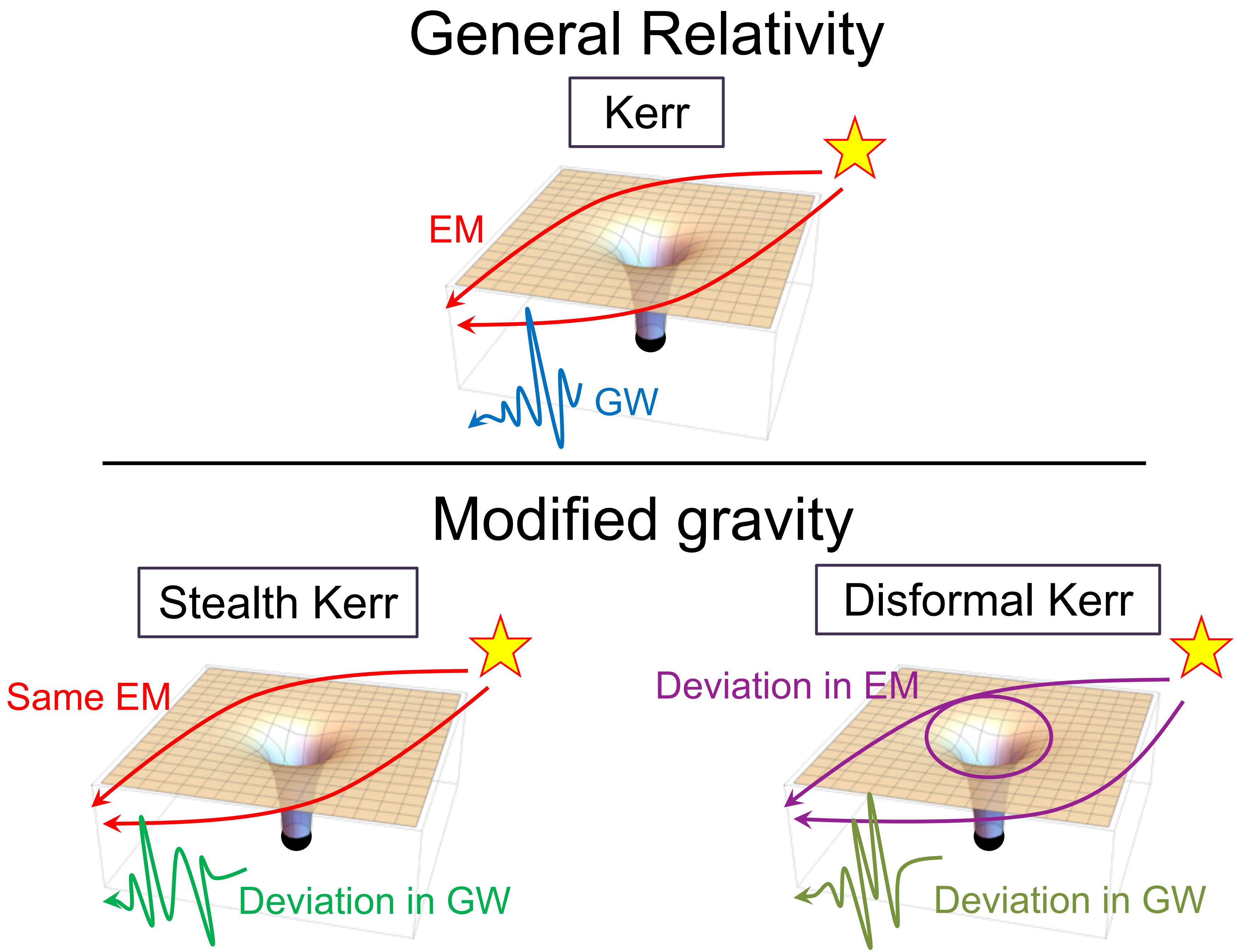
Vishveshwara (1980)

General Relativity

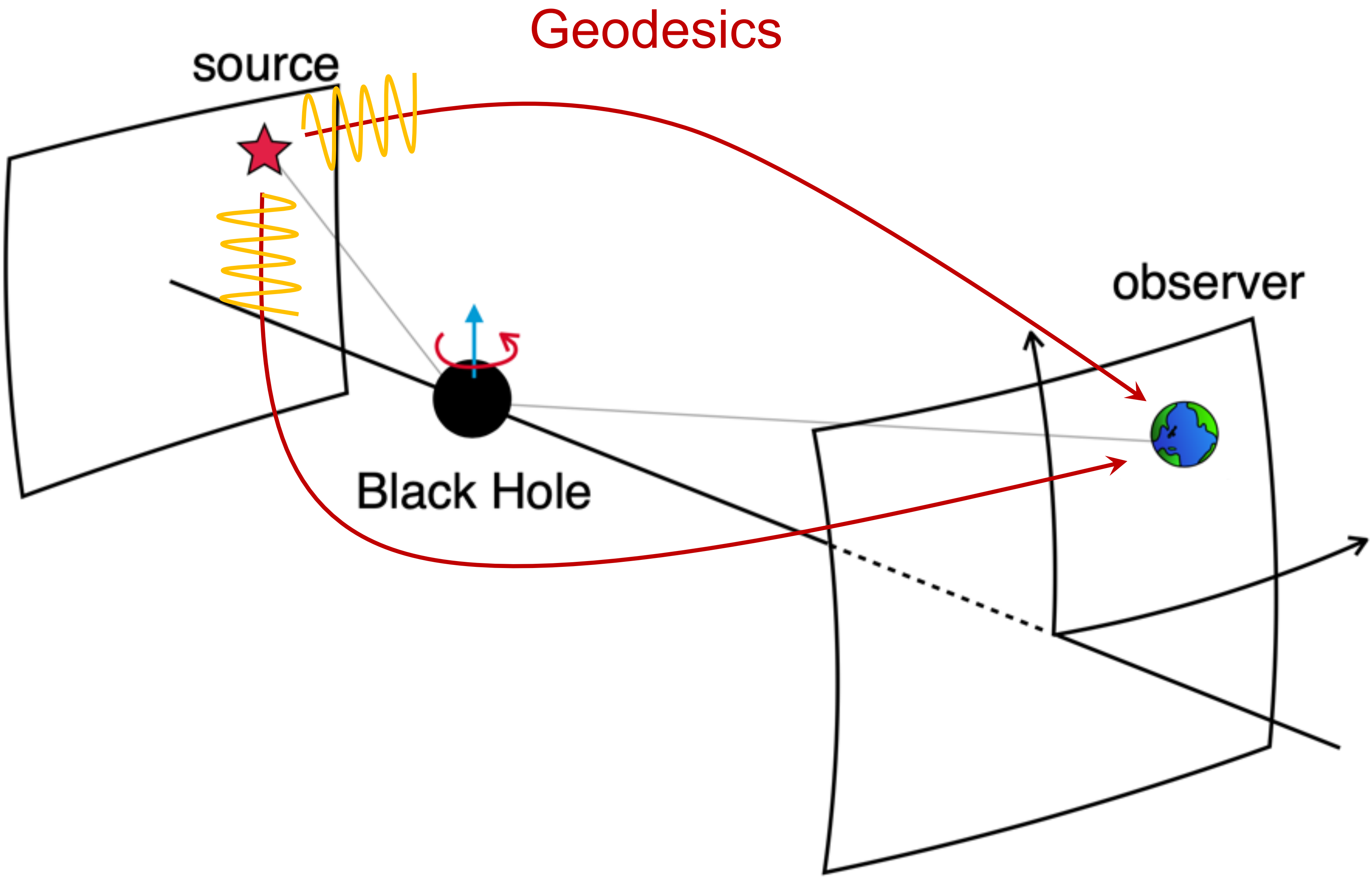
$\boxed{\text{Kerr solution}}$

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

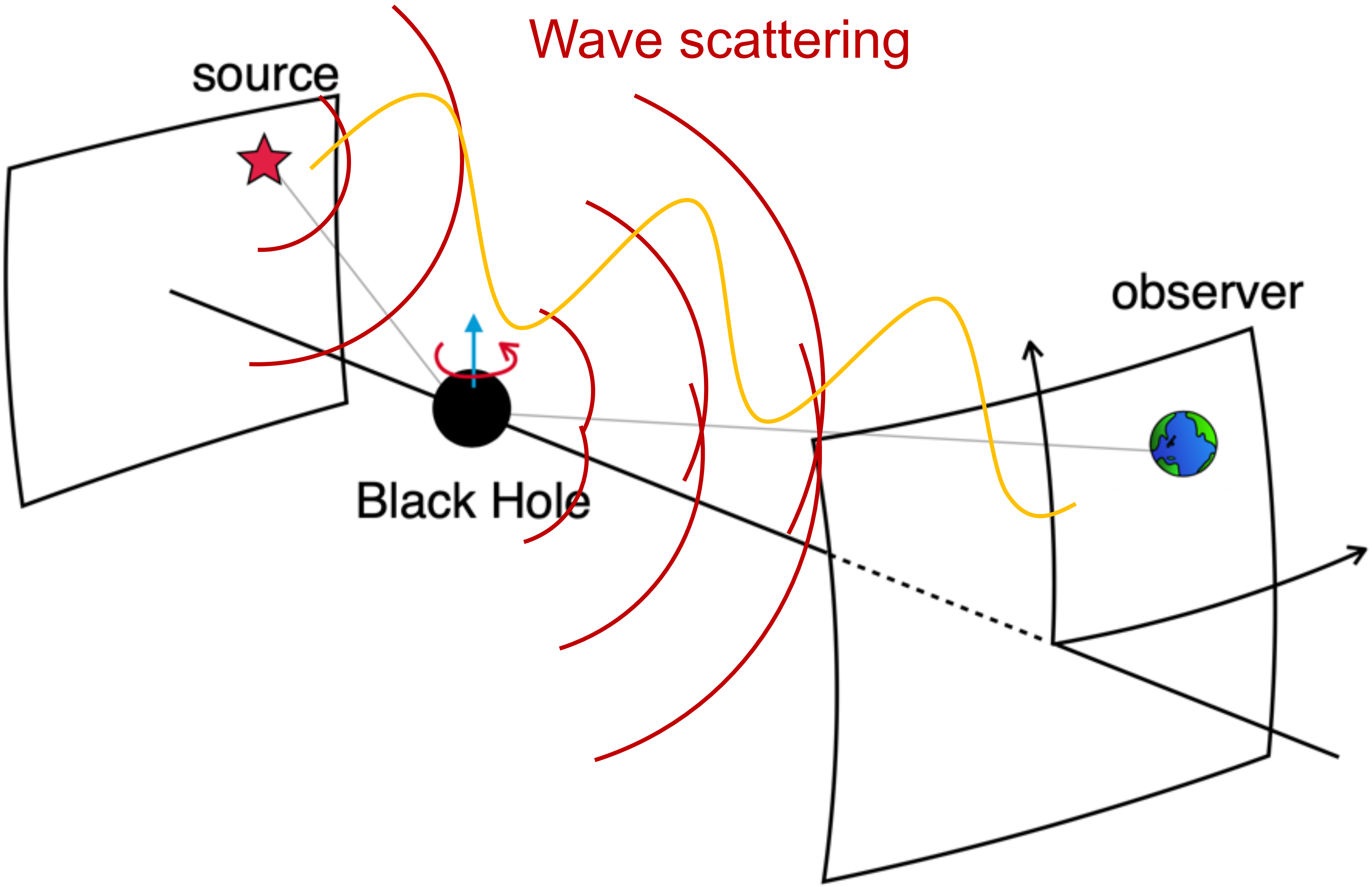



General Relativity
Kerr
EM
GW
Modified gravity
Stealth Kerr
Same EM
Deviation in GW
Disformal Kerr
Deviation in EM
Deviation in GW

Geometrical optics



Wave optics



Scattering from black holes

- While one can numerically integrate Teukolsky equation, analytic treatment is also important.
- There are many analytic works using some approximations in the literature (e.g. WKB).
- We establish an exact formulation w/o any approximations for wave scattering of spin-s waves from Kerr-Newman-de Sitter BH.

- Teukolsky

$$\left[\Delta^{-s}\frac{d}{dr}\Delta^{s+1}\frac{d}{dr}+\frac{J^2-isJ\Delta'}{\Delta}+2isJ'\right.$$
$$\left.-\frac{2}{3}\Lambda r^2(s+1)(2s+1)+2s(1-\alpha)-\lambda\right]R_s=0$$

- Schrödinger

$$\left(\frac{d^2}{dr_*^2}+V_s\right)\mathcal{Y}_s=0$$

- Heun

$$y''+\left(\frac{\gamma}{z}+\frac{\delta}{z-1}+\frac{\epsilon}{z-a}\right)y'$$
$$+\frac{\alpha\beta z-q}{z(z-1)(z-a)}y=0$$

- Teukolsky

$$\left[\Delta^{-s}\frac{d}{dr}\Delta^{s+1}\frac{d}{dr}+\frac{J^2-isJ\Delta'}{\Delta}+2isJ' \right.$$
$$\left.-\frac{2}{3}\Lambda r^2(s+1)(2s+1)+2s(1-\alpha)-\lambda\right]R_s=0$$

Saul Teukolsky
(1947–)

- Heun

- Teukolsky

Erwin Schrödinger
(1887–1961)

- Schrödinger

$$\left(\frac{d^2}{dr_*^2} + V_s\right) \mathcal{Y}_s = 0$$

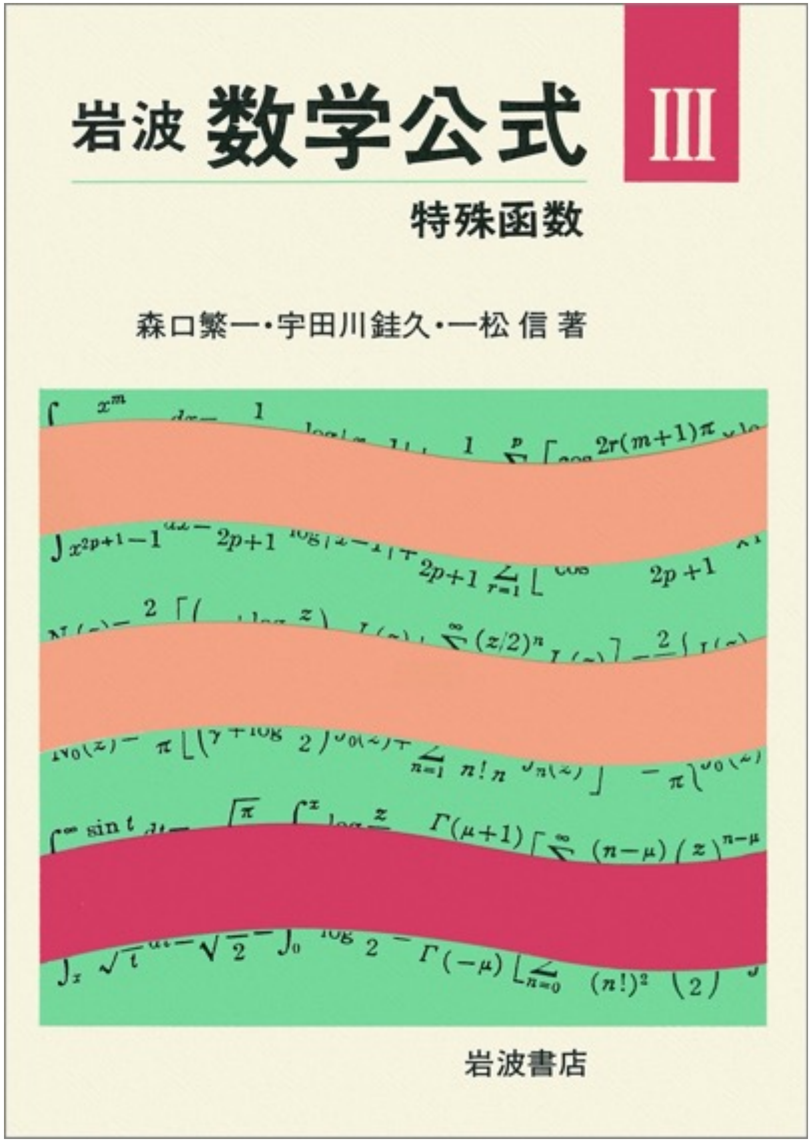
Kahl Heun (1859–1929)

- Heun

$$y'' + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a}\right) y' + \frac{\alpha\beta z - q}{z(z-1)(z-a)} y = 0$$

Pronunciation of “Heun” follows German rule – it’s just like ‘coin’ or ‘join’.

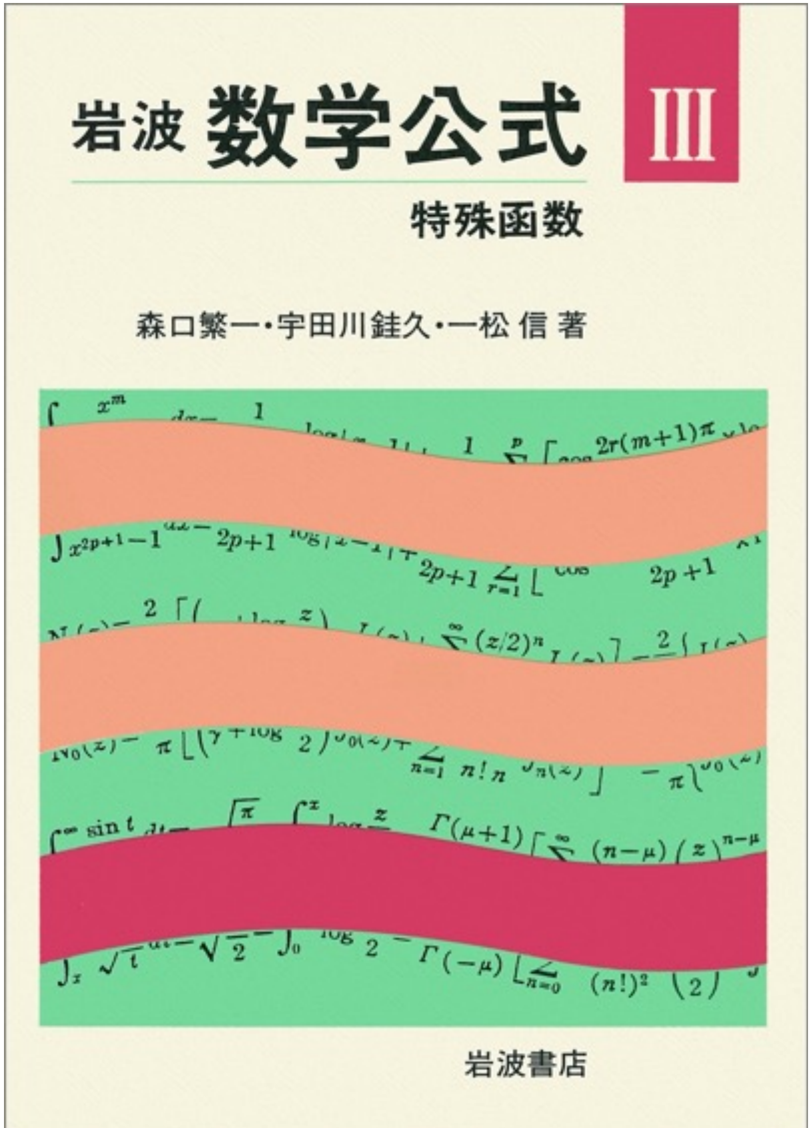
Iwanami Mathematical formulas III: Special functions



Contents:

1. $\Gamma(z), \zeta(z)$, integrals
2. Elliptic function
3. Hypergeometric function
4. $P_n(x), H_n(x)$ etc
5. $P_\nu(z), Q_\nu(z), Y_{\ell m}(\theta, \phi)$
6. $J_\nu(z), N_\nu(z), j_n(z), n_n(z)$
7. Lame, Mathieu functions

Iwanami Mathematical formulas III: Special functions



Contents:

1. $\Gamma(z), \zeta(z)$, integrals
2. Elliptic function

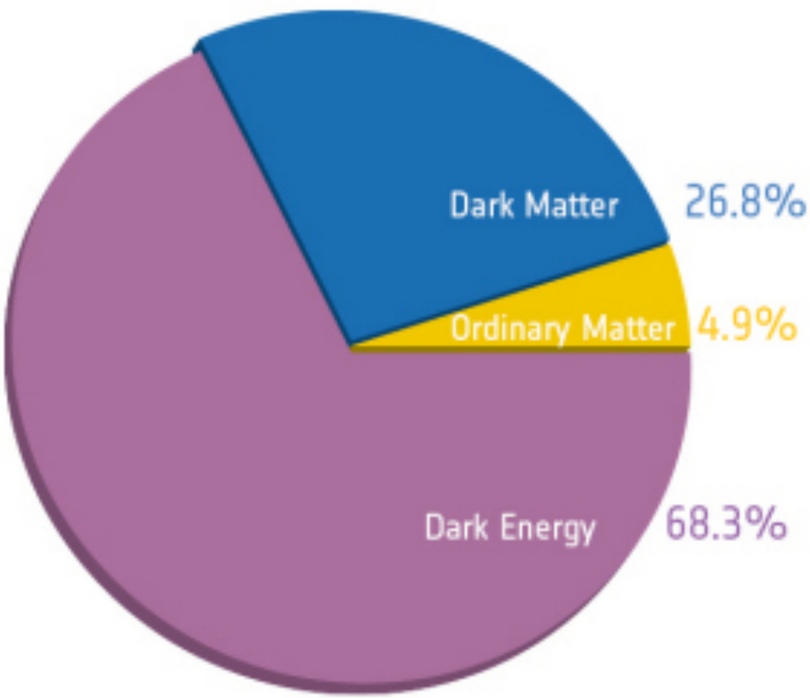
22%

70%

3. Hypergeometric function
4. $P_n(x), H_n(x)$ etc
5. $P_\nu(z), Q_\nu(z), Y_{\ell m}(\theta, \phi)$
6. $J_\nu(z), N_\nu(z), j_n(z), n_n(z)$
7. Lame, Mathieu functions

Heun 8%

Iwanami Mathematical formulas III: Special functions



Contents:

1. $\Gamma(z), \zeta(z)$, integrals
2. Elliptic function

22%

70%

3. Hypergeometric function
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6. $J_\nu(z), N_\nu(z), j_n(z), n_n(z)$
7. Lame, Mathieu functions

Heun 8%

Iwanami Mathematical formulas III: Special functions

Contents:

1. $\Gamma(z), \zeta(z)$, integrals
2. Elliptic function

22%

70%

3. Hypergeometric function
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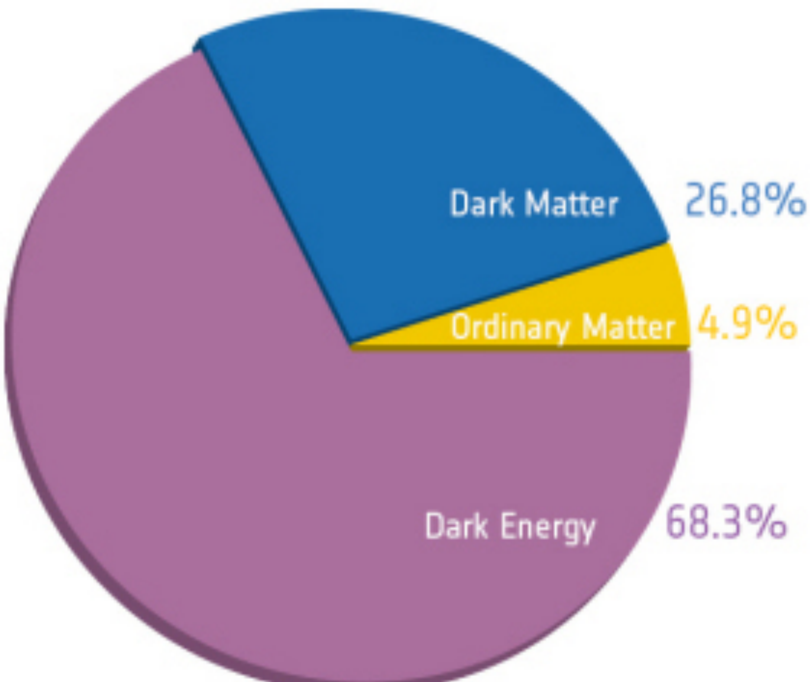
6. $J_\nu(z), N_\nu(z), j_n(z), n_n(z)$
7. Lame, Mathieu functions

Heun 78%

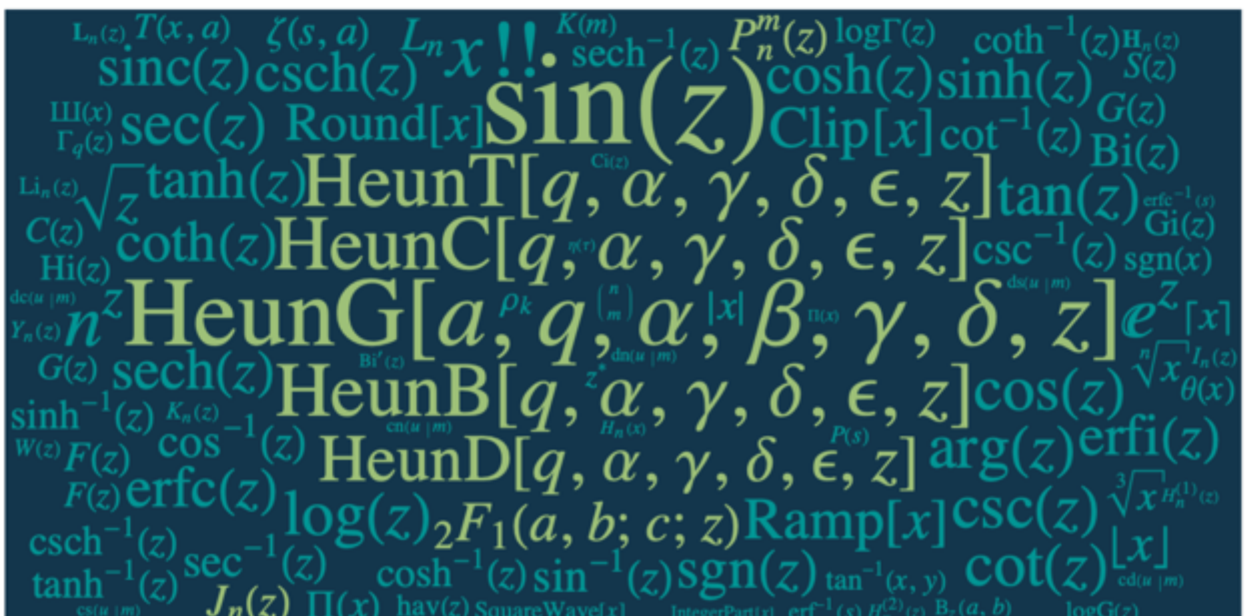
Strictly speaking, most functions are special case of Heun.



From Sine to Heun: 5 New Functions for Mathematics and Physics in the Wolfram Language

May 6, 2020 — Tigran Ishkhanyan, Algorithms R&D

Mathematica was initially built to be a universal solver of different mathematical tasks for everything from school-level algebraic equations to complicated problems in real scientific projects. During the past 30 years of development, over 250 mathematical functions have been implemented in the system, and in the recent release of Version 12.1 of the Wolfram Language, we've added many more, ranging from the elementary Sin function to the advanced Heun functions.



These and a lot of other interesting examples on the properties and applications of the Heun functions are noted in the documentation pages.

Heun Functions in Physics Heun functions have a range of applications in contemporary physics and are powerful enough to generate solutions for a significant set of unsolved problems from quantum mechanics, the theory of black holes, conformal field theory and others. They are being successfully applied in real physical problems at a rapid rate: during the last decade, the number of publications related to the theory of Heun functions tripled in comparison with all other publications until 2010, according to arXiv.

Specifically, the powerful apparatus of the Heun functions allows derivation of new infinite classes of integrable potentials for relativistic and nonrelativistic wave equations used in different problems of quantum control and engineering (please see the recent paper by A. M. Ishkhanyan for different examples).

Heun functions appear in the theory of Kerr–de Sitter black holes and may be used for analysis in more complex geometries (the papers by R. S. Borissov and P. P. Fiziev and H. Suzuki, E. Takasugi and H. Umetsu discuss these problems).

The relationship between the Heun class of equations and Painlevé transcendents leads to new results for the two-dimensional conformal field theory based on the analysis of the solutions of Heun equations (see the papers of B. C. da Cunha and J. P. Cavalcante and F. Atai and E. Langmann).

The aforementioned examples as well as others indicate that the Heun functions are important in and popular for solving absolutely different problems in contemporary physics.

Closing Words At Wolfram, we are in a constant search for fresh ideas and methods that make the Wolfram Language one of the most famous, popular, powerful and user-friendly tools for

References

- Futterman, Handler, Matzner, “Scattering from Black Holes”, (Cambridge Univ. Press, 1988)
- Ronveaux, “Heun’s Differential Equations”, (Oxford Univ. Press, 1995)
- Nakamura, Oohara, Kojima, PTP Supp. 1987
- Suzuki, Takasugi, Umetsu, PTP 1998, 1999, 2000
- Hatsuda, 2006.08957

“STU” in this talk

Contents

Kerr-Newman-de Sitter spacetime

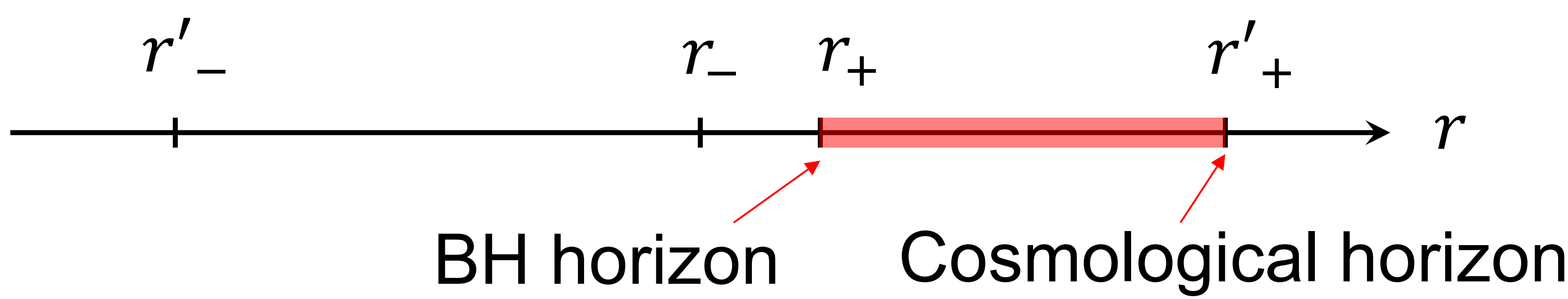
$$ds^2 = -\rho^2\left(\frac{dr^2}{\Delta} + \frac{d\theta}{1+\alpha\cos^2\theta}\right) - \frac{(1+\alpha\cos^2\theta)\sin^2\theta}{(1+\alpha)^2\rho^2}(adt-\rho^2 d\varphi)^2 + \frac{\Delta}{(1+\alpha)^2\rho^2}(dt - a\sin^2\theta d\varphi)^2$$

where

$$\Delta(r) = (r^2+a^2)\left(1-\frac{\Lambda r^2}{3}\right) - 2Mr + Q^2, \quad \alpha = \frac{\Lambda a^2}{3}, \quad \rho^2 = r^2 + a^2$$

$\Delta(r) = 0$ has four roots:

$$\Delta(r) = -\frac{\Lambda}{3}(r-r_-)(r-r_+)(r-r'_-)(r-r'_+)$$



Teukolsky equation

- We consider a massless test field with spin s on KNdS background.
- The Teukolsky equation is then separable and can be transformed into the Heun equation. STU98

(Actually this holds for spin s field on general type-D vacuum background with CC.)

Batic, Schmid, gr-qc/0701064

- We decompose the master variables in NP formalism as

$$\psi_s = R_s(r) S_s(\theta) e^{-i\omega t + im\varphi}$$

Angular Teukolsky equation

$$\left[\frac{d}{dx}(1+\alpha x^2)(1-x^2)\frac{d}{dx}+\lambda-s(1-\alpha)-2\alpha x^2\right.$$
$$\left.+\frac{4sx(1+\alpha)[m\alpha-c(1+\alpha)]}{1+\alpha x^2}-\frac{(1+\alpha)^2[m+sx-(1-x^2)c]^2}{(1+\alpha x^2)(1-x^2)}\right]S_s=0$$

where $x=\cos\theta$, $c=a\omega$.

Solution for SdS or RNdS:
spin-weighted spherical harmonics

$$S_s(\theta)e^{im\varphi}={}_sY_{\ell m}(\theta,\varphi)$$
$$\lambda=\ell(\ell+1)-s(s-1)$$

Angular Teukolsky equation

$$\left[\frac{d}{dx}(1+\alpha x^2)(1-x^2)\frac{d}{dx}+\lambda-s(1-\alpha)-2\alpha x^2 + \frac{4sx(1+\alpha)[m\alpha-c(1+\alpha)]}{1+\alpha x^2}-\frac{(1+\alpha)^2[m+sx-(1-x^2)c]^2}{(1+\alpha x^2)(1-x^2)}\right]S_s=0$$

where $x=\cos\theta$, $c=a\omega$.

$(1+\alpha x^2)(1-x^2)=0$ has four roots.

$\Longrightarrow$ Four regular singular points: ± 1, $\pm i/\sqrt{\alpha}$

$\Longrightarrow$ Heun equation

(∞ is removable singularity)

Radial Teukolsky equation

$$\left[\Delta^{-s}\frac{d}{dr}\Delta^{s+1}\frac{d}{dr}+\frac{J^2-isJ\Delta'}{\Delta}+2isJ'\right.$$
$$\left.-\frac{2}{3}\Lambda r^2(s+1)(2s+1)+2s(1-\alpha)-\lambda\right]R_s=0$$

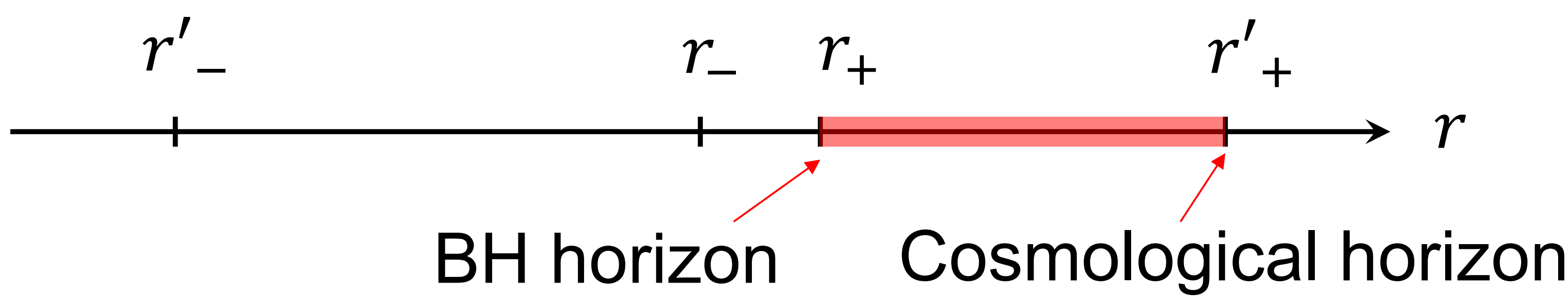
where $J(r)=(1+\alpha)[\omega(r^2+a^2)-am]-eQr$.

$\Delta(r)=0$ has four roots.

$\Longrightarrow$ Four regular singular points: r_-, r_+, r'_-, r'_+

$\Longrightarrow$ Heun equation

(∞ is removable singularity)



- Teukolsky

$$\left[\Delta^{-s}\frac{d}{dr}\Delta^{s+1}\frac{d}{dr}+\frac{J^2-isJ\Delta'}{\Delta}+2isJ'\right.$$
$$\left.-\frac{2}{3}\Lambda r^2(s+1)(2s+1)+2s(1-\alpha)-\lambda\right]R_s=0$$

- Heun

$$y''+\left(\frac{\gamma}{z}+\frac{\delta}{z-1}+\frac{\epsilon}{z-a}\right)y'$$
$$+\frac{\alpha\beta z-q}{z(z-1)(z-a)}y=0$$

- Schrödinger

$$\left(\frac{d^2}{dr_*^2}+V_s\right)\mathcal{Y}_s=0$$

Hypergeometric equation

$$y'' + \left(\frac{\gamma}{z} + \frac{\delta}{z-1}\right)y' + \frac{\alpha\beta}{z(z-1)}y = 0$$

$$\delta = \alpha + \beta - \gamma + 1$$

- Three regular singular points: $0, 1, \infty$
- Three indep. parameters: α, β, γ
- Hypergeometric function $F(\alpha, \beta, \gamma; z)$

Heun equation

$$y'' + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a}\right) y' + \frac{\alpha\beta z - q}{z(z-1)(z-a)} y = 0$$

$$\epsilon = \alpha + \beta - \gamma - \delta + 1, \qquad a \neq 0, 1$$

- Four regular singular points: $0, 1, a, \infty$

- Six indep. parameters

a : singularity parameter

$\alpha, \beta, \gamma, \delta(, \epsilon)$: exponent parameters

q : accessory parameter

Heun equation

$$y'' + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a}\right) y' + \frac{\alpha\beta z - q}{z(z-1)(z-a)} y = 0$$

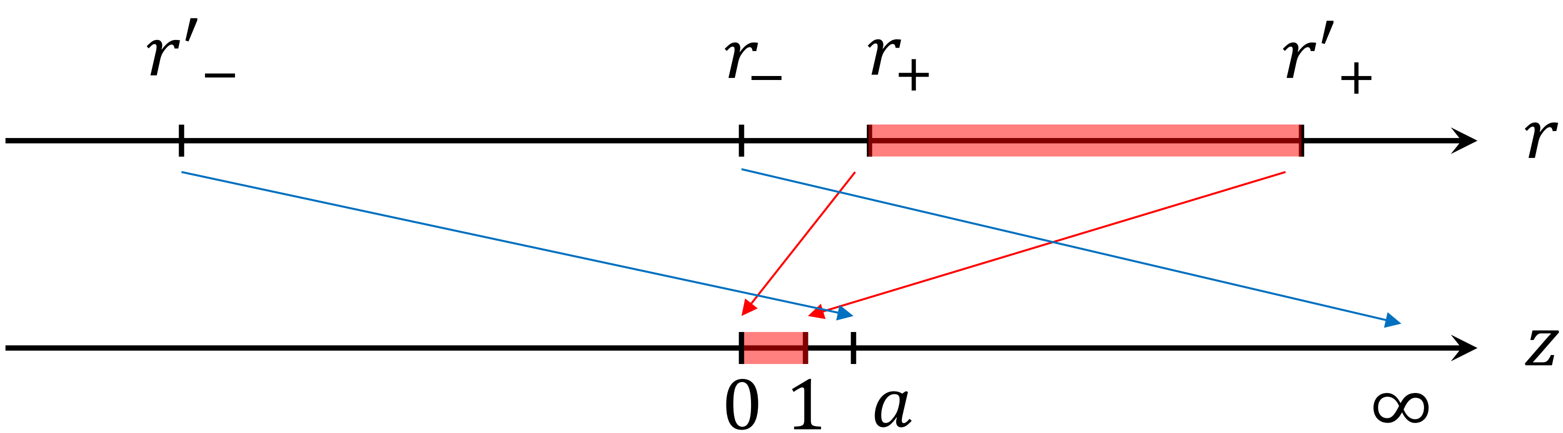
$$\epsilon = \alpha + \beta - \gamma - \delta + 1, \qquad a \neq 0, 1$$

- $4! = 24$ transformations

  We choose a transformation such that

$$(r'_-, r_-, r_+, r'_+) \to (a, \infty, 0, 1)$$

cf. STU
$(\infty, 0, 1, a)$
$(\infty, 1, 0, a)$
Hatsuda
$(a, \infty, 0, 1)$



Teukolsky → Heun

$$\left[\Delta^{-s}\frac{d}{dr}\Delta^{s+1}\frac{d}{dr}+\frac{J^2-isJ\Delta'}{\Delta}+2isJ'\right.$$
$$\left.-\frac{2}{3}\Lambda r^2(s+1)(2s+1)+2s(1-\alpha)-\lambda\right]R_s=0$$

Transformation

$$z=\frac{r'_+-r_-}{r'_+-r_+}\frac{r-r_+}{r-r_-}$$
$$R_s(r)=z^{B_1}(z-1)^{B_2}(z-z_r)^{B_3}(z-z_\infty)^{2s+1}y_s(z)$$

where $z_\infty=z|_{r\to\infty},\ z_r=z|_{r\to r'_-}$.

We also define $B(r)=iJ(r)/\Delta'(r)$ and

$B_1=B(r_+)$, $B_2=B(r'_+)$, $B_3=B(r'_-)$, $B_4=B(r_-)$.

Teukolsky → Heun

$$y'' + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a}\right)y' + \frac{\alpha\beta z - q}{z(z-1)(z-a)}y = 0$$

$$\epsilon = \alpha + \beta - \gamma - \delta + 1, \qquad a \neq 0, 1$$

with

$$a = z_r, q = -v(\lambda, s, \Lambda), \alpha = 2s + 1, \beta = s + 1 - 2B_4,$$
$$\gamma = 2B_1 + s + 1, \delta = 2B_2 + s + 1, \epsilon = 2B_3 + s + 1$$

Note that $a = z_r > 1$.

Heun equation

$$y'' + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a}\right) y' + \frac{\alpha\beta z - q}{z(z-1)(z-a)} y = 0$$

$$\epsilon = \alpha + \beta - \gamma - \delta + 1, \qquad a \neq 0, 1$$

Several types of exact solutions

- Local solution or local Heun function Hl
- Heun function Hf
- Heun polynomial Hp
- Path-multiplicative solutions
- Series of hypergeometric functions

Simplest but small region of convergence. We use this.

Wider region of convergence but more involved. STU98,99,00

Convergence of c_n^ν

$$y_\nu(z) = \sum_{n=-\infty}^{\infty} c_n^\nu F(-\nu, \nu + \omega; \gamma; z)$$

Three-term recurrence relation

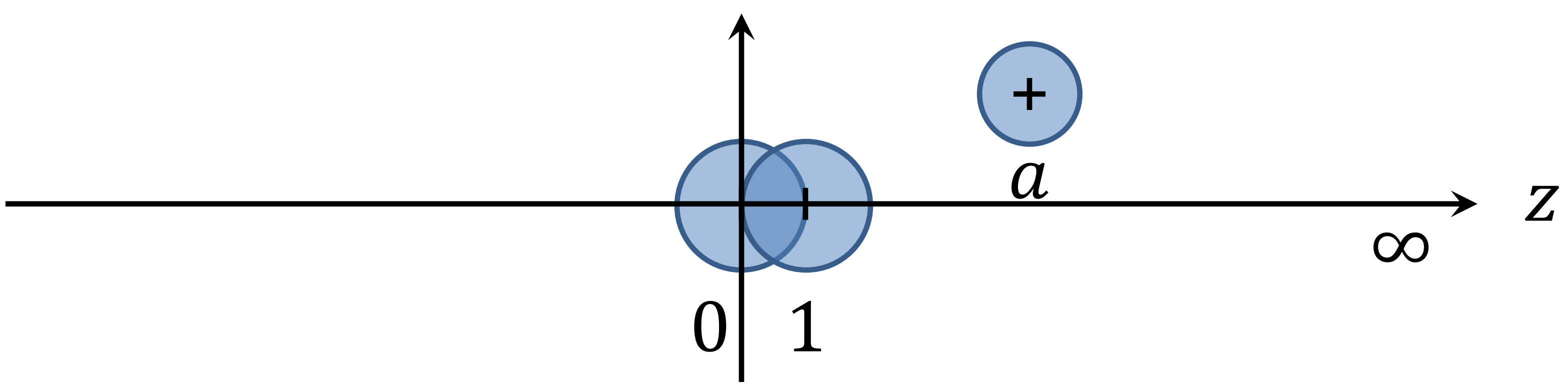
Heun equation

$$y'' + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a}\right)y' + \frac{\alpha\beta z - q}{z(z-1)(z-a)}y = 0$$

$$\epsilon = \alpha + \beta - \gamma - \delta + 1, \qquad a \neq 0, 1$$

- Local Heun function (Frobenius solution) Hl

Analytic around a regular singularity.



Hypergeometric function

$$F(\alpha, \beta, \gamma; z) = \sum_{k=0}^{\infty} c_k z^k$$

- Recurrence relation

$$c_0 = 1$$
$$(k+1)(k+\gamma)c_{k+1} - (k+\alpha)(k+\beta)c_k = 0$$

- Solution

$$F(\alpha, \beta, \gamma; z) = 1 + \frac{\alpha\beta}{\gamma 1!} z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)2!} z^2 + \cdots$$
$$= \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+k)\Gamma(\beta+k)}{\Gamma(\gamma+k)} \frac{z^k}{k!}$$

Local Heun function

$$Hl(a,q;\alpha,\beta,\gamma,\delta;z)=\sum_{k=0}^{\infty}c_k z^k$$

- Three-term recurrence relation

$$\begin{aligned}&c_{-1}=0,\quad c_0=1\\&(k+1)(k+\gamma)ac_{k+1}\\&-\{k[(k+\gamma+\delta-1)a+(k+\gamma+\epsilon-1)]+q\}c_k\\&+(k+\alpha-1)(k+\beta-1)c_{k-1}=0\end{aligned}$$

No simple expression for general c_k.

- Radius of convergence $=\min(1,|a|)$
- $Hl(1,\alpha\beta;\alpha,\beta,\gamma,\delta;z)=F(\alpha,\beta,\gamma;z)$
- $Hl=$ “HeunG” in Mathematica 12.1 or later.

Local Heun functions at $z = 0{,}1$

- Local solutions at $z = 0$ (BH horizon)

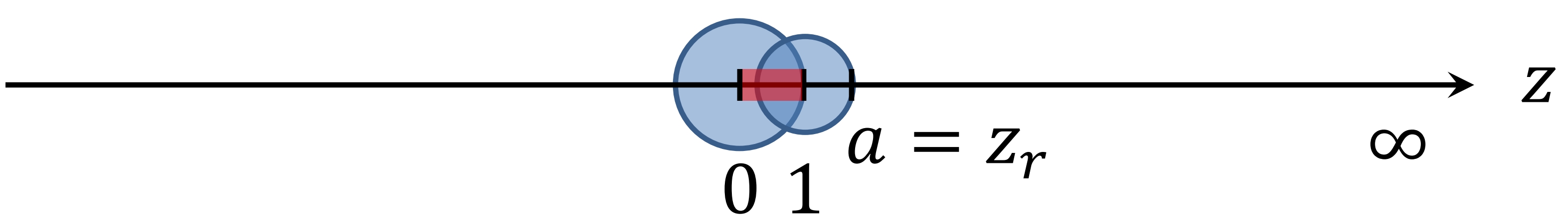
$$y_{01}(z) = Hl(a, q; \alpha, \beta, \gamma, \delta; z)$$
$$y_{02}(z) = z^{1-\gamma} Hl(a, q_2; \alpha_2, \beta_2, \gamma_2, \delta_2; z)$$

- Local solutions at $z = 1$ (Cosmological horizon)

$$y_{11}(z) = Hl(1-a, \alpha\beta - q; \alpha, \beta, \gamma, \delta; 1-z)$$
$$y_{12}(z) = (1-z)^{1-\delta} Hl\left(1-a, q'_2; \alpha'_2, \beta'_2, \gamma'_2, \delta'_2; 1-z\right)$$

$|a| > 1$ so there is an overlapping region of the two convergence circles.

$\Longrightarrow$ In this region we can use both local solutions w/o analytic continuation.



Connection coefficients

Relation between the two sets of local solutions

$$y_{01}(z) = C_{11} y_{11}(z) + C_{12} y_{12}(z)$$
$$y_{02}(z) = C_{21} y_{11}(z) + C_{22} y_{12}(z)$$

No useful analytic expressions are known for C_{ij}

but we can obtain them by e.g. $\longleftrightarrow$ cf. Gauss formula for F

$$C_{22} = \frac{W_z[y_{02}, y_{11}]}{W_z[y_{12}, y_{11}]}$$

where $W_z[u, v] = u\frac{dv}{dz} - \frac{du}{dz}v$ is the Wronskian.

Note that $W_z[y_a, y_b] \neq \text{const.}$ but

$$z^{\gamma}(z-1)^{\delta}(z-a)^{\epsilon} W_z[y_a, y_b] = \text{const.}$$

Connection coefficients

Similarly,

$$y_{11}(z) = D_{11}y_{01}(z) + D_{12}y_{02}(z)$$
$$y_{12}(z) = D_{21}y_{01}(z) + D_{22}y_{02}(z)$$

where

$$\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}^{-1}$$
$$= \frac{W_z[y_{11}, y_{12}]}{W_z[y_{01}, y_{02}]} \begin{pmatrix} C_{22} & C_{12} \\ -C_{21} & C_{11} \end{pmatrix}$$

Angular part

- ✓ Similarly, we can also transform the angular Teukolsky equation into the Heun form.

- ✓ Requirement of the regularity at $\theta = 0$ and π determines the eigenvalue λ :

$$W_z\left[y_{a,0I}, y_{a,1J}\right] = 0$$

with $I = \begin{cases} 1 \ (m - s \geq 0) \\ 2 \ (m - s < 0) \end{cases}$ and $J = \begin{cases} 1 \ (m + s \leq 0) \\ 2 \ (m + s > 0) \end{cases}$.

- Teukolsky

$$\left[\Delta^{-s}\frac{d}{dr}\Delta^{s+1}\frac{d}{dr}+\frac{J^2-isJ\Delta'}{\Delta}+2isJ'\right.$$
$$\left.-\frac{2}{3}\Lambda r^2(s+1)(2s+1)+2s(1-\alpha)-\lambda\right]R_s=0$$

- Heun

$$y''+\left(\frac{\gamma}{z}+\frac{\delta}{z-1}+\frac{\epsilon}{z-a}\right)y'$$
$$+\frac{\alpha\beta z-q}{z(z-1)(z-a)}y=0$$

- Schrödinger

$$\left(\frac{d^2}{dr_*^2}+V_s\right)\mathcal{Y}_s=0$$

Contents

- Teukolsky

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Schrödinger equation

$$\left(\frac{d^2}{dr_*^2} + V_s\right)\mathcal{Y}_s = 0$$

where $dr_* = \rho^2 dr/\Delta$, $\mathcal{Y}_s = \Delta^{s/2}\rho R_s$ and

$$V_s(r) = \frac{\Delta'^2}{\rho^4}\left(B + \frac{s}{2}\right)^2 + \frac{\Delta}{\rho^4}[\dots]$$

Pure imaginary

s^2, is

$\Longrightarrow V_s(r) = V_{-s}^*(r)$

$\Longrightarrow \mathcal{Y}_s$ and $\mathcal{Y}_{-s}^*$ are solutions

$\Longrightarrow R_s$ and $\Delta^{-s} R_{-s}^*$ are solutions

Schrödinger equation

$$\left(\frac{d^2}{dr_*^2} + V_s\right)\mathcal{Y}_s = 0$$

For $r \to r_+$ or r'_+

$$V_s(r) \to \frac{\Delta'^2}{\rho^4}\left(B + \frac{s}{2}\right)^2 + \cancel{\frac{\Delta}{\rho^4}[\dots]}$$

$$= \frac{\Delta'^2}{\rho^4}\left(B + \frac{s}{2}\right)^2$$

Schrödinger equation

$$\left(\frac{d^2}{dr_*^2} + V_s\right)\mathcal{Y}_s = 0$$

For $r \to r_+$ or r'_+

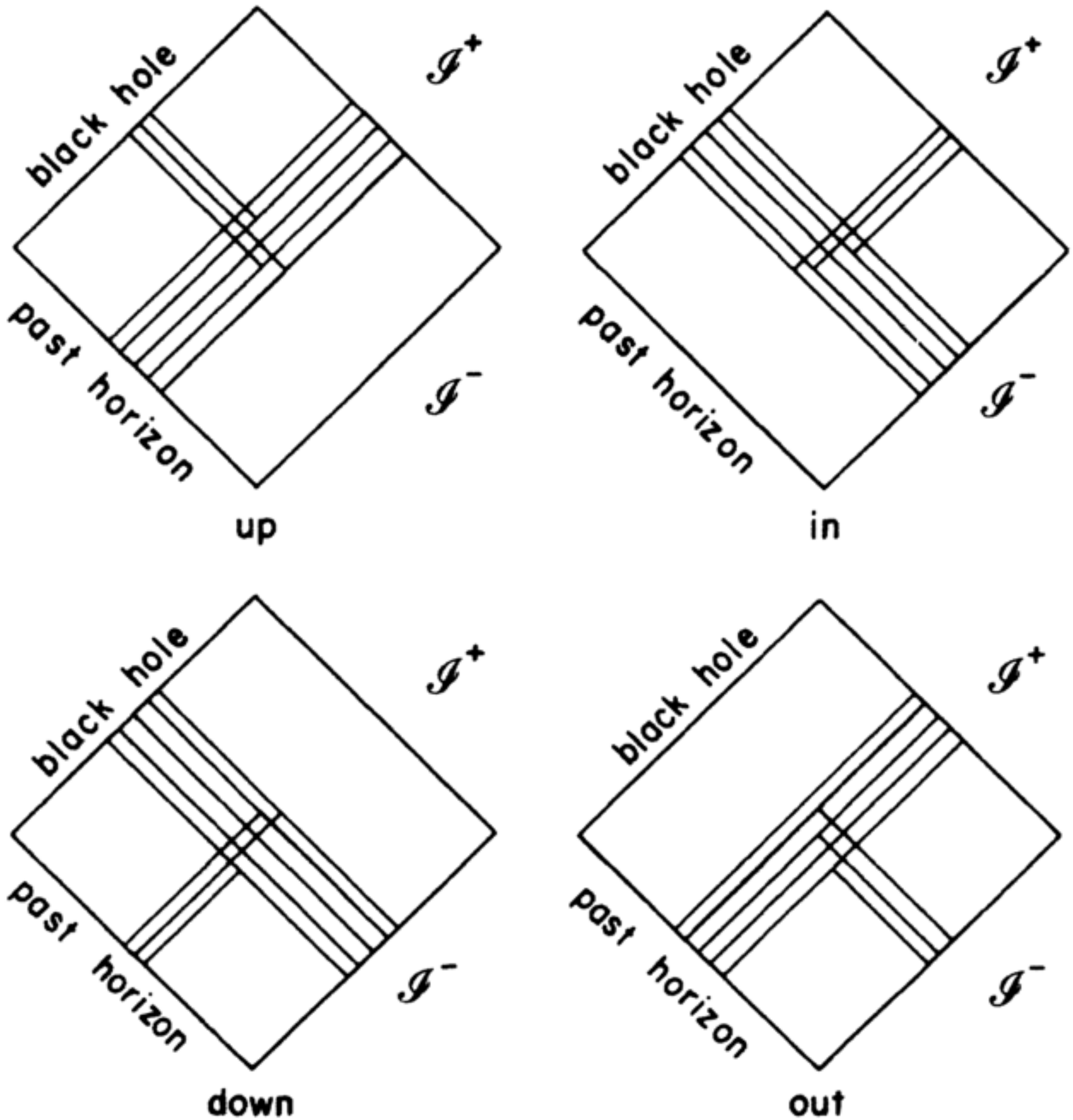
$$\mathcal{Y}_s \to \exp\left[\pm\frac{\Delta'}{\rho^2}\left(B + \frac{s}{2}\right)r_*\right]$$

and hence

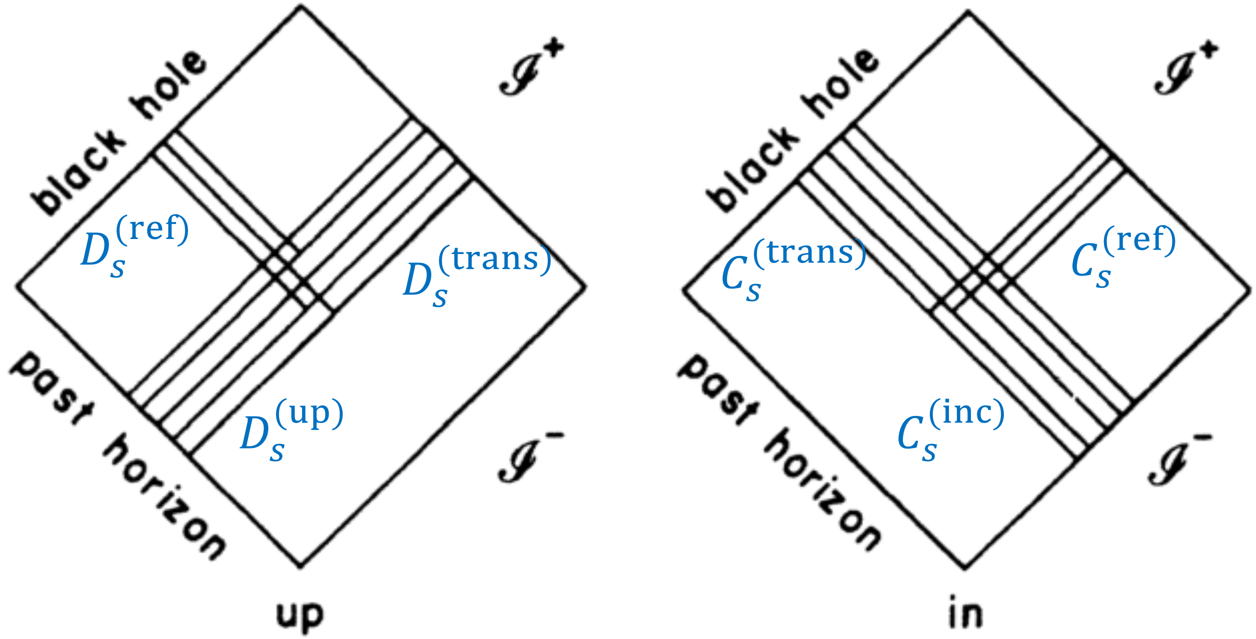
$$R_s \to \Delta^B \text{ and } \Delta^{-B-s} \quad (e^{i\omega r_*} \text{ and } \Delta^{-s}e^{-i\omega r_*})$$

The asymptotic behavior of general solution is thus a linear combination of $e^{i\omega r_*}$ and $e^{-i\omega r_*}$.

$\times e^{-i\omega t}$ Outgoing Ingoing



Chrzanpwski, Misner (1974)



Boundary condition

$$R_{\text{in},s} \to \begin{cases} C_s^{(\text{trans})} \Delta^{-B_1 - s}, & (r \to r_+) \\ C_s^{(\text{ref})} \Delta^{B_2} + C_s^{(\text{inc})} \Delta^{-B_2 - s}, & (r \to r'_+) \end{cases}$$

$$R_{\text{up},s} \to \begin{cases} D_s^{(\text{up})} \Delta^{B_1} + D_s^{(\text{ref})} \Delta^{-B_1 - s}, & (r \to r_+) \\ D_s^{(\text{trans})} \Delta^{B_2}, & (r \to r'_+) \end{cases}$$

- Teukolsky

$$\left[\Delta^{-s}\frac{d}{dr}\Delta^{s+1}\frac{d}{dr}+\frac{J^2-isJ\Delta'}{\Delta}+2isJ'\right.$$
$$\left.-\frac{2}{3}\Lambda r^2(s+1)(2s+1)+2s(1-\alpha)-\lambda\right]R_s=0$$

- Schrödinger

$$\left(\frac{d^2}{dr_*^2}+V_s\right)\mathcal{Y}_s=0$$

- Heun

$$y''+\left(\frac{\gamma}{z}+\frac{\delta}{z-1}+\frac{\epsilon}{z-a}\right)y'$$
$$+\frac{\alpha\beta z-q}{z(z-1)(z-a)}y=0$$

- **Teukolsky**

$$\left[\Delta^{-s}\frac{d}{dr}\Delta^{s+1}\frac{d}{dr}+\frac{J^2-isJ\Delta'}{\Delta}+2isJ' -\frac{2}{3}\Lambda r^2(s+1)(2s+1)+2s(1-\alpha)-\lambda\right]R_s=0$$

- **Schrödinger**

$$\left(\frac{d^2}{dr_*^2}+V_s\right)\mathcal{Y}_s=0$$

- **Heun**

$$y''+\left(\frac{\gamma}{z}+\frac{\delta}{z-1}+\frac{\epsilon}{z-a}\right)y'+\frac{\alpha\beta z-q}{z(z-1)(z-a)}y=0$$

Asymptotic behavior of exact solution

Let's identify $R_{\mathrm{in},s}$ and $R_{\mathrm{up},s}$ in terms of the exact solution.

- Local solutions at $z = 0$ (BH horizon)

$$y_{01}(z) = 1 + O(z)$$
$$y_{02}(z) = z^{-2B_2-s}[1 + O(z)]$$

- Local solutions at $z = 1$ (Cosmological horizon)

$$y_{11}(z) = 1 + O(1 - z)$$
$$y_{12}(z) = (1 - z)^{-2B_3-s}[1 + O(1 - z)]$$

Asymptotic behavior of exact solution

After some calculations we see that

$$R_{\mathrm{in},s} = \begin{cases} R_{02,s}, & (r \to r_+) \\ C_{21} R_{11,s} + C_{22} R_{12,s}, & (r \to r'_+) \end{cases}$$

$$R_{\mathrm{up},s} = \begin{cases} D_{11} R_{01,s} + D_{12} R_{02,s}, & (r \to r_+) \\ R_{11,s}, & (r \to r'_+) \end{cases}$$

Each of two expressions equal exactly.

We can then obtain $C_s^{(\mathrm{inc})}$ etc in terms of $C_{22,s}$ etc.

e.g. $C_s^{(\mathrm{inc})} \propto C_{22,s}$

$\Longrightarrow$ We can calculate $C_s^{(\mathrm{inc})}$ etc exactly.

Contents

Quasinormal mode

We can derive QNM frequency by

Hatsuda, 2006.08957

$$W_z\left[y_{a,0I}, y_{a,1J}\right] = 0 \ \text{ and } \ C_{22} = 0$$

Pros:

Simple and fast (a few sec).

No approximation.

Easy to increase the accuracy.

Cons:

Requires an initial value $(\omega_{\mathrm{ini}}, \lambda_{\mathrm{ini}})$ close to the correct ones.

cf. - Leaver's continued fraction method: Similar but it includes infinite series, which one needs to truncate and check the convergence (while the conv. is fast).

- Numerical calculation: More processes which make it difficult to control the accuracy.

Reflection / transmission rate

We obtained a simple exact formula

$$\mathcal{R}_s = 1 - \mathcal{T}_s$$

$$\mathcal{T}_s = F_s \left(\frac{z_\infty}{z_\infty - 1}\right)^2 \frac{1}{C_{22,s} C^*_{22,-s}}$$

with $F_s = \frac{\Delta'(r_+)(2B_1+s)}{\Delta'(r'_+)(2B_2+s)}$.

cf. - STU00 formula: Similar but it includes infinite series.

- Numerical calculation: More processes (solve tortoise coord., fit, impose boundary condition with shooting method), difficult to control the accuracy.

Pros:

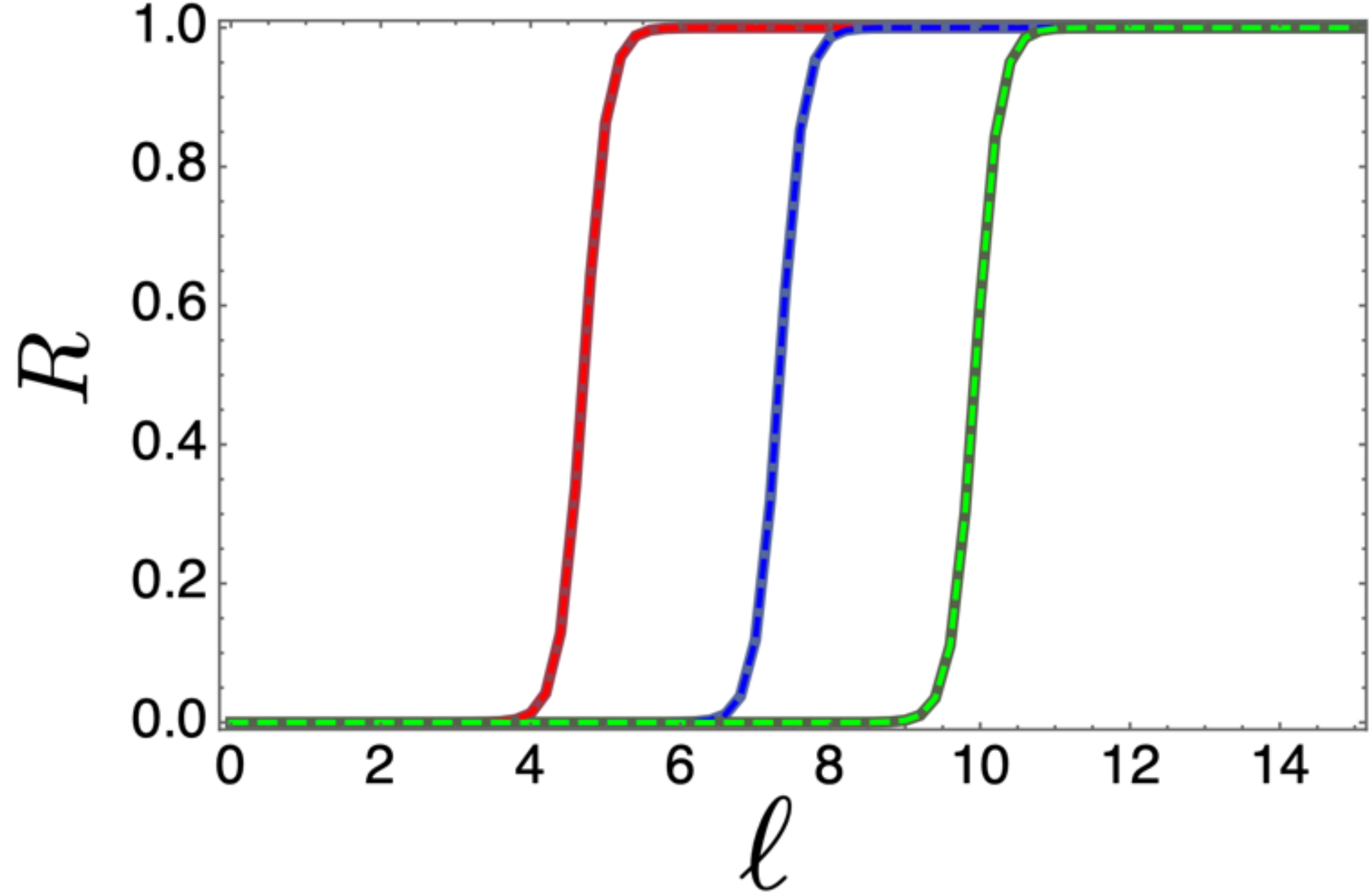
Simple and fast.

No approximation.

Easy to increase the accuracy.

Reflection rate

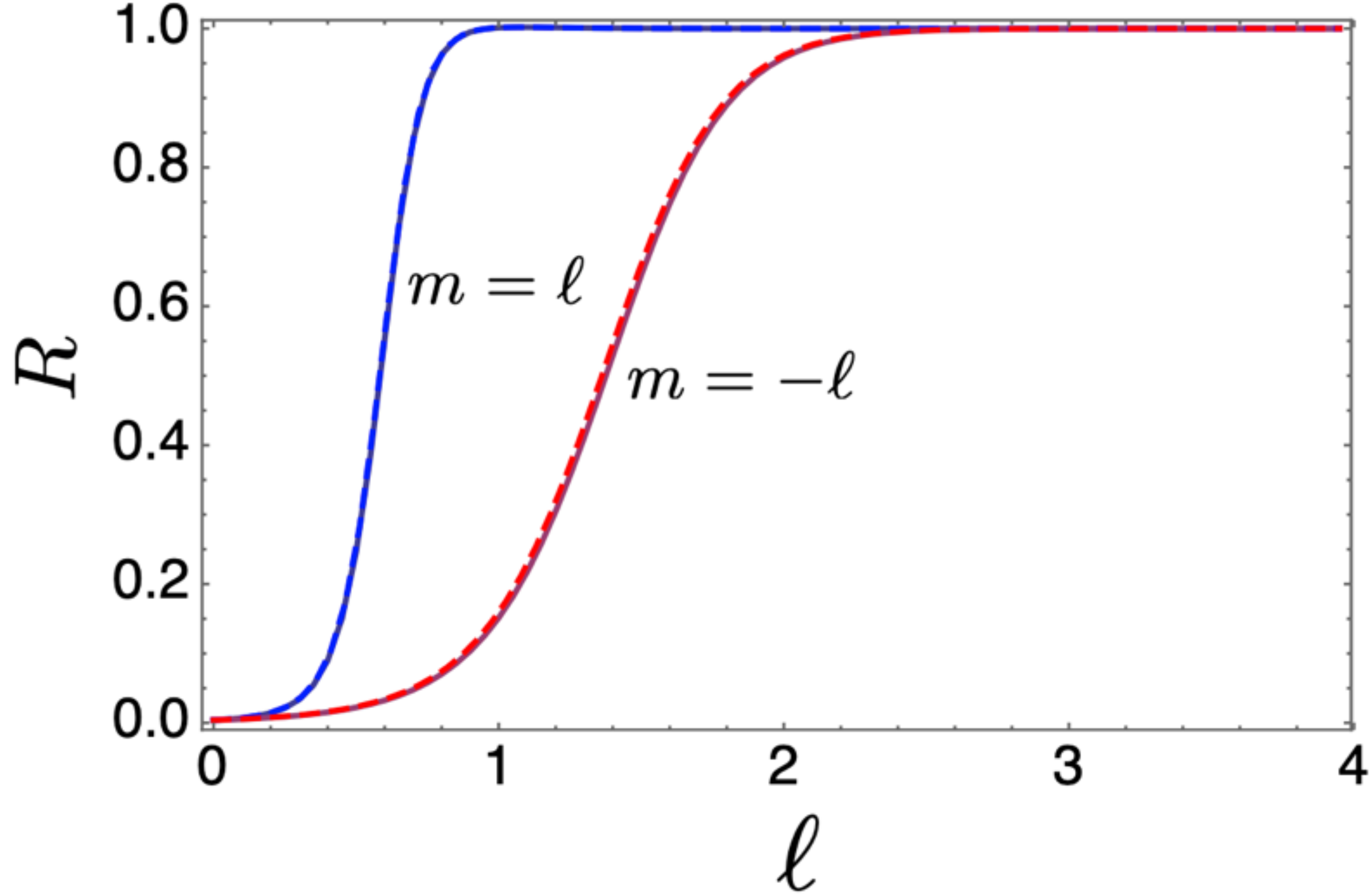
Schwarzschild-de Sitter $\Lambda M^2 = 10^{-3}$
$s = 0, \omega M = 1, 1.5, 2$



Our exact formula (~10sec, WorkingPrecision → 20)
Numerical calculation (~1s, MachinePrecision)

Reflection rate

Kerr-de Sitter $\Lambda M^2 = 10^{-3}, a/M = 0.9$
$s = 0, \omega M = 0.3$

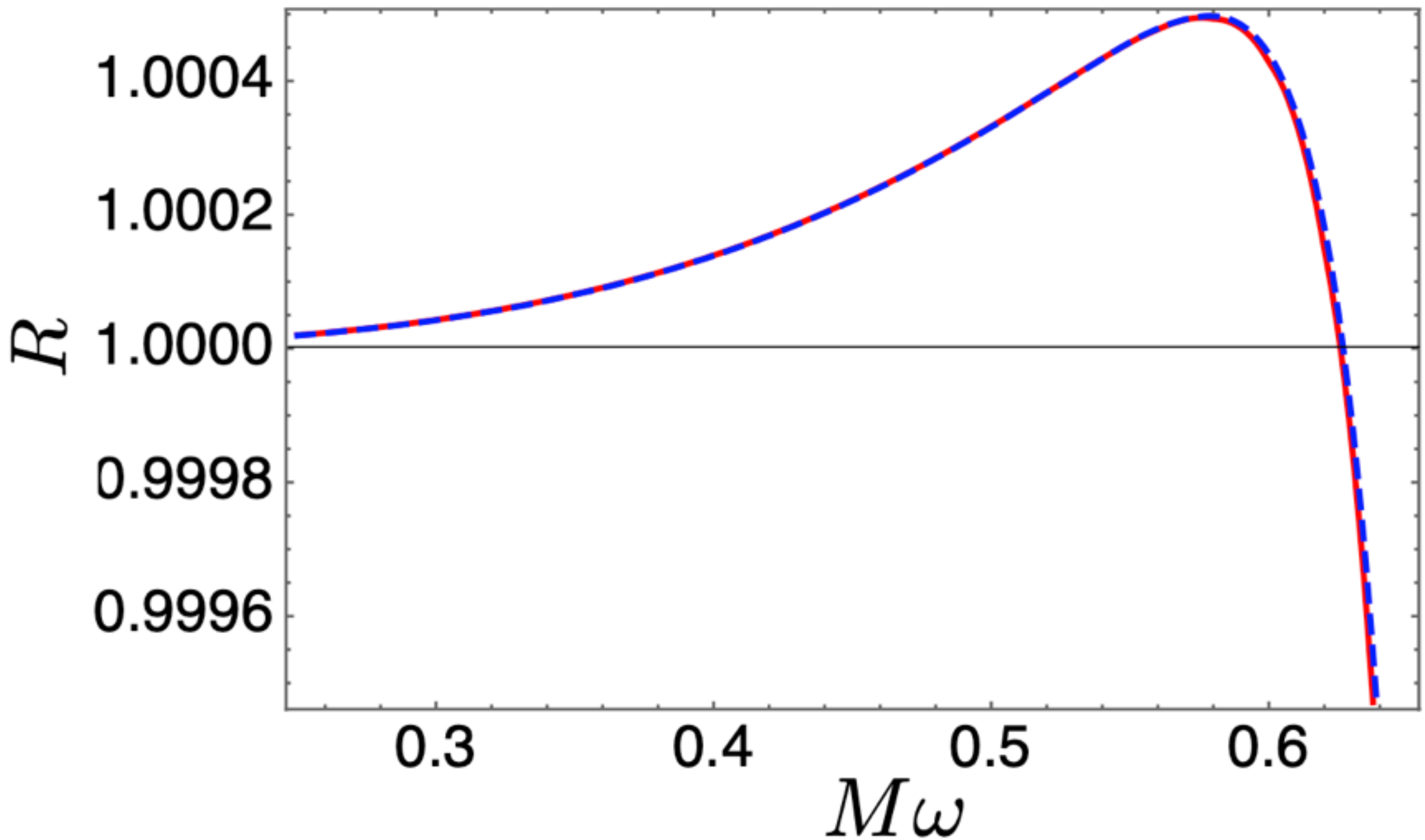


Our exact formula

Numerical integration with shooting method

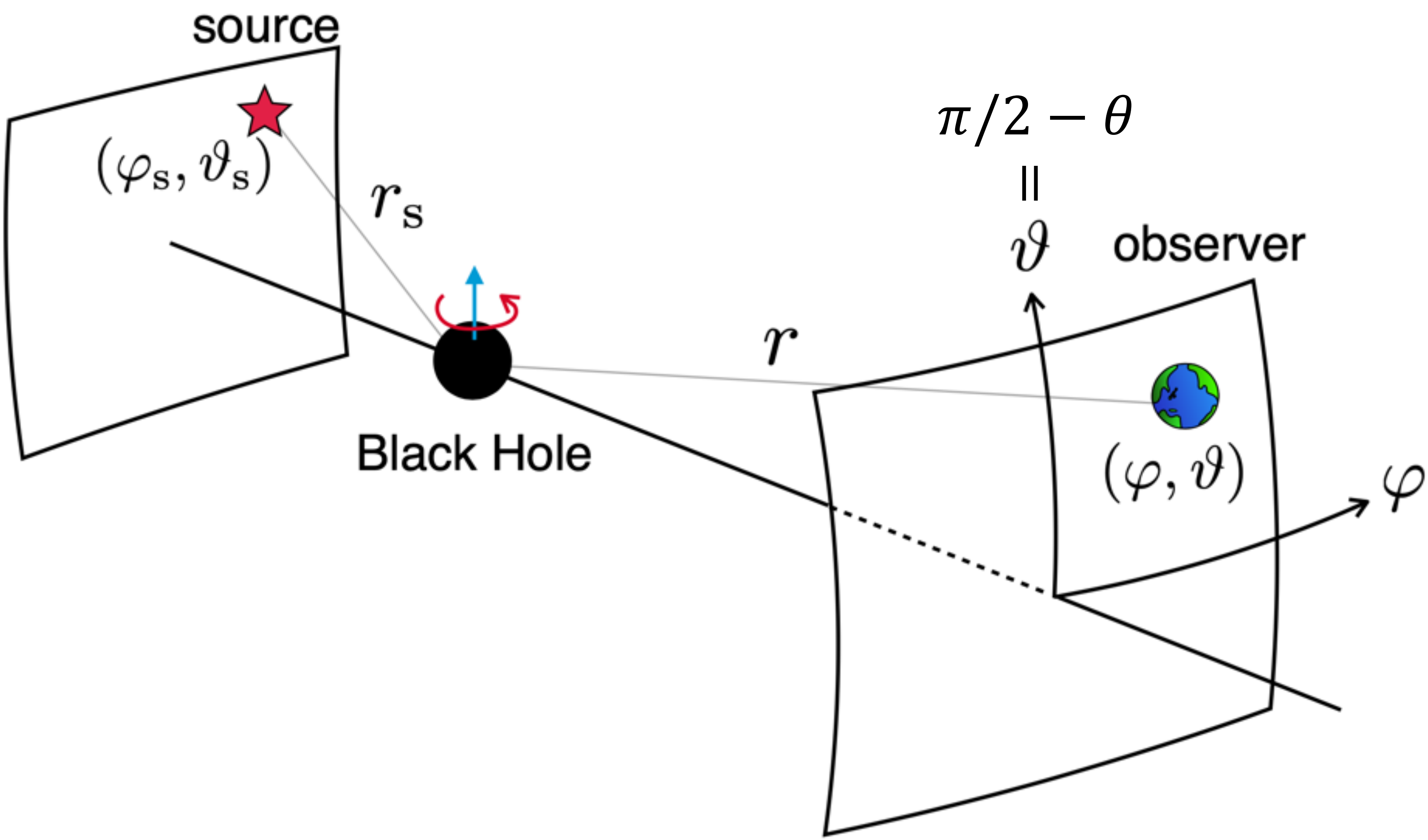
Superradiance

Kerr-de Sitter $\Lambda M^2 = 10^{-3}, a/M = 0.9$
$s = 0, \ell = m = 2$



Our exact formula

Numerical integration with shooting method



source
(φ_s, ϑ_s)
r_s
Black Hole
r
$\pi/2 - \theta$
$\|$
ϑ
observer
(φ, ϑ)
φ

Green function

We constructed the Green function

$$G(\boldsymbol{x}, \boldsymbol{x}_\mathrm{s}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{-\Delta^s(r_\mathrm{s})\left[R_{\mathrm{in},s}(r_\mathrm{s}) R_{\mathrm{up},s}(r_\mathrm{s}) \Theta(r - r_\mathrm{s}) + (r \leftrightarrow r_\mathrm{s})\right]}{\Delta^{s+1} W_r\left[R_{\mathrm{in},s}, R_{\mathrm{up},s}\right]} \times {}_sY_{\ell\mathrm{m}}(\theta, \varphi)\, {}_sY^*_{\ell\mathrm{m}}(\theta_\mathrm{s}, \varphi_\mathrm{s})$$

Pros:

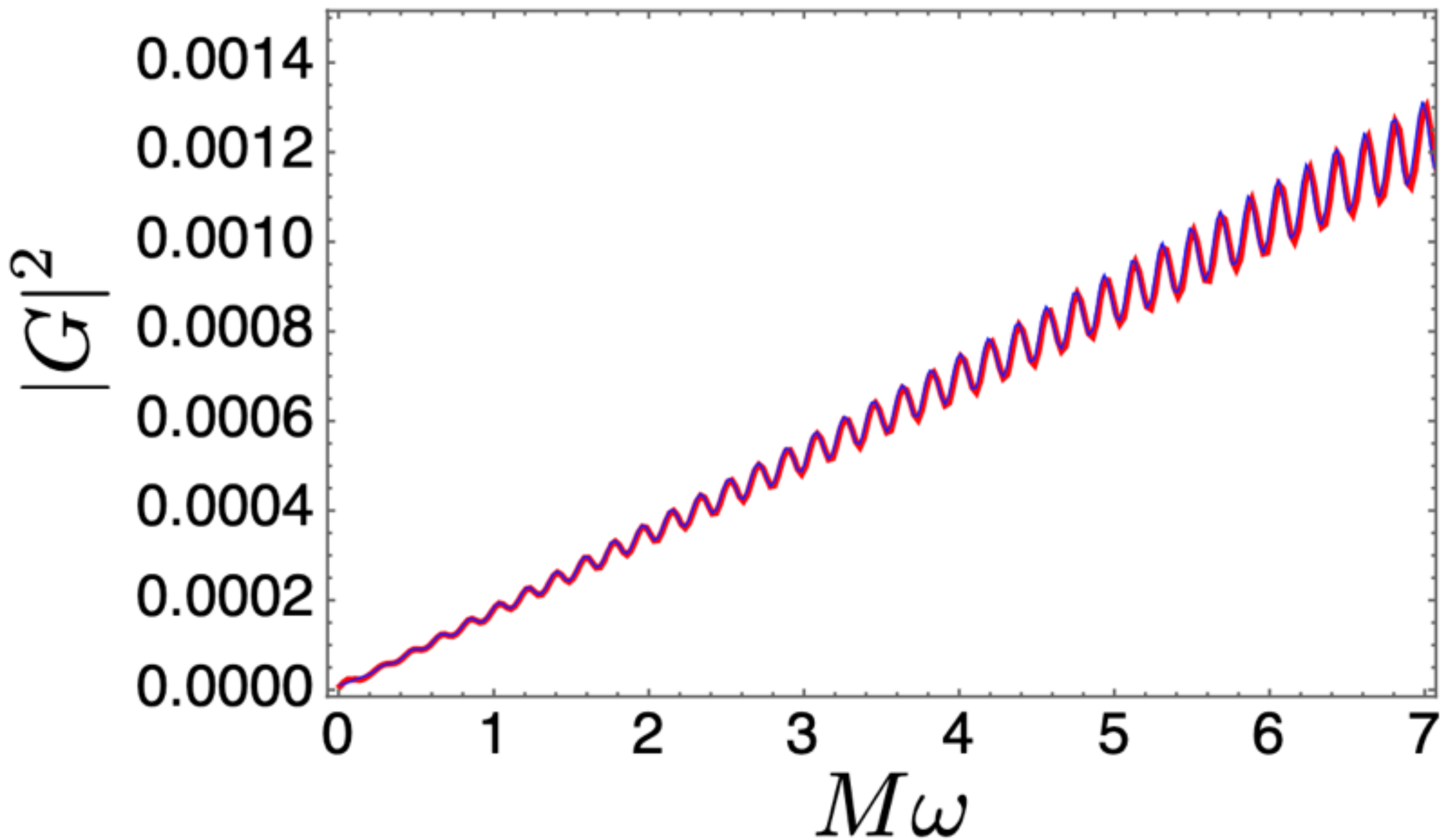
No approximation such as

- $r, r_\mathrm{s} \gg 1$
- $\vartheta, \varphi \ll 1$
- $\omega M \gg 1$ or $\omega M \ll 1$

cf.
Nambu, Noda, 1502.05468
Nambu, Noda, Sakai, 1905.01793

Forward scattering

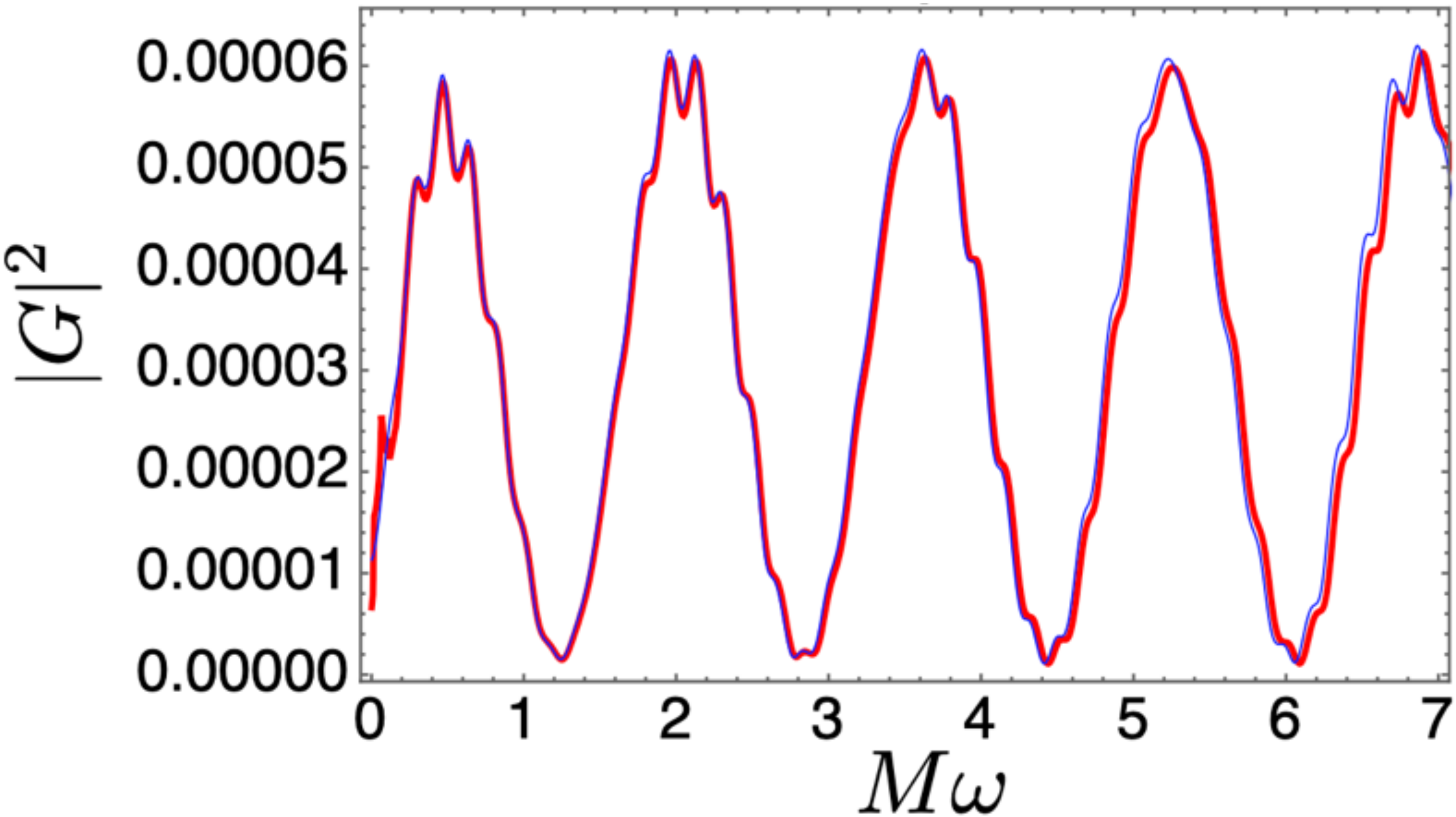
$s = 0,$
$(r, \vartheta, \varphi) = (20M, 0,0)$
$(r_s, \vartheta_s, \varphi_s) = (6M, 0, \pi)$



Our exact formula for SdS with $\Lambda M^2 = 10^{-3}$
Numerical integration for Schwarzschild

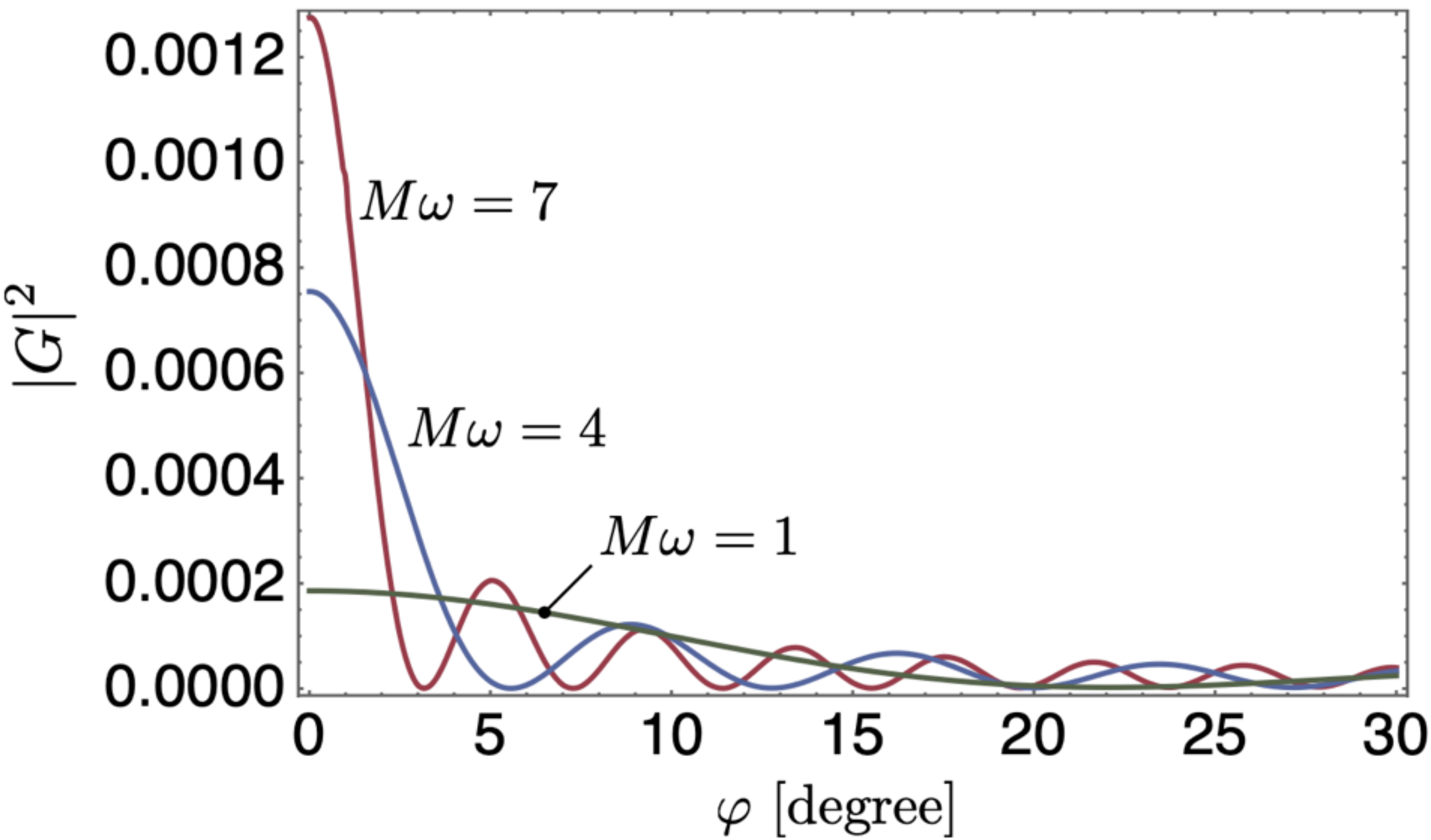
Slightly off forward scattering

$s = 0,$
$(r, \vartheta, \varphi) = (20M, 0, \pi/10)$
$(r_s, \vartheta_s, \varphi_s) = (6M, 0, \pi)$



Our exact formula for SdS with $\Lambda M^2 = 10^{-3}$
Numerical integration for Schwarzschild

Angular dependence



Contents

Summary

- We have established the exact formulation of the scattering problem of spin-s massless field on KNdS background by using the exact solution in terms of local Heun function Hl.
- Simple and fast formulae w/o any approximations.
- Applications include:

  - QNM
  - Reflection / absorption rates
  - Green function
  - S-matrix, cross section, BH image, …and more!