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Generalized 3D-reconstruction method of dipolar and quadripolar anisotropies in cosmic-ray distributions

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We present a method to reconstruct an underlying dipole and quadripole in the angular distribution of cosmic-rays, extending the standard Rayleigh method by using both the right ascension and the declination of the arrival directions, and generalizing the full-sky coverage method of Sommers (2001) to partial-sky coverage experiments. The statistical properties of the method are described and the concept of amplitude and angular reconstruction power is introduced to quantify the efficiency of the method.

1. Introduction

To second order of the spherical harmonic expansion, the CR flux from direction $\mathbf{u}$ can be written:

$$\Phi(\mathbf{u}) = \frac{\Phi_0}{4\pi}(1 + \alpha\mathbf{D}\cdot\mathbf{u} + \sum_{i,j} Q_{ij}u_iu_j) = \frac{\Phi_0}{4\pi}(1 + \alpha\mathbf{D}\cdot\mathbf{u} + \sum_{i,j}\lambda_i(\mathbf{q_i}\cdot\mathbf{u})^2), \tag{1}$$

which consists of an isotropic part, $\Phi_0/4\pi$, modulated by a dipolar component characterized by its direction, $\mathbf{D}$, and amplitude, α ($0 \leq \alpha \leq 1$), and a quadripolar component described by a traceless and symmetric 2nd-order tensor, Q, which has five independent components related to the $l = 2$ spherical harmonics coefficients $a_{l,m}$. The symmetric real matrix, Q_{ij}, can be diagonalised in an orthogonal basis. Let λ_i and $\mathbf{q_i}$ be its three eigenvalues and unit eigenvectors. The quadripole amplitude, β, can be defined by $\beta \equiv \frac{\Phi_{max}-\Phi_{\min}}{\Phi_{\max}+\Phi_{\min}} = \frac{\lambda_+-\lambda_-}{2+\lambda_++\lambda_-}$, where λ_+ is the largest positive eigenvalue (in direction $\mathbf{q}_+$) and λ_- the minimum eigenvalue (in direction $\mathbf{q}_-$). The five independent parameters are then two amplitudes (β,λ_+) and three angles: (θ_+, φ_+) defining the orientation of $\mathbf{q}_+$ and φ_- defining the orientation of $\mathbf{q}_-$ in the plan perpendicular to $\mathbf{q}_+$.

2. Description of the method

The idea of the method is to compute some integral quantities and to identify their theoretical values with the discrete versions provided by discrete sums over experimental data. To reconstruct the three parameters of the dipole and the five parameters of a general quadripole, we need eight quantities and an additional one for the global flux normalization. This is provided by the following moments of order zero, one and two:

$$I_0 = \int \Phi(\mathbf{u})d\Omega \quad , \quad \mathbf{I} = \int \mathbf{u}\Phi(\mathbf{u})d\Omega \quad \text{and} \quad I_{ij} = \int \Phi(\mathbf{u})(u_iu_j - \frac{1}{3}\delta_{ij})d\Omega, \tag{2}$$

which can be integrated over the whole sky (in equatorial coordinates), with $\Phi(\mathbf{u})$ as in Eq. (1):

$$I_0 = \Phi_0 \qquad \mathbf{I} = \frac{1}{3}\Phi_0 \times \alpha\mathbf{D} \quad , \quad \text{and} \quad I_{ij} = \frac{2}{15}\Phi_0 \times Q_{ij} \tag{3}$$

Discrete versions of these integrals can be obtained by summing the corresponding quantity in sky pixels and rewriting the sums over solid angle as sums of cosmic-ray (CR) arrival directions, $\mathbf{u}_\mathrm{k}$, with $1 \leq k \leq N$, where

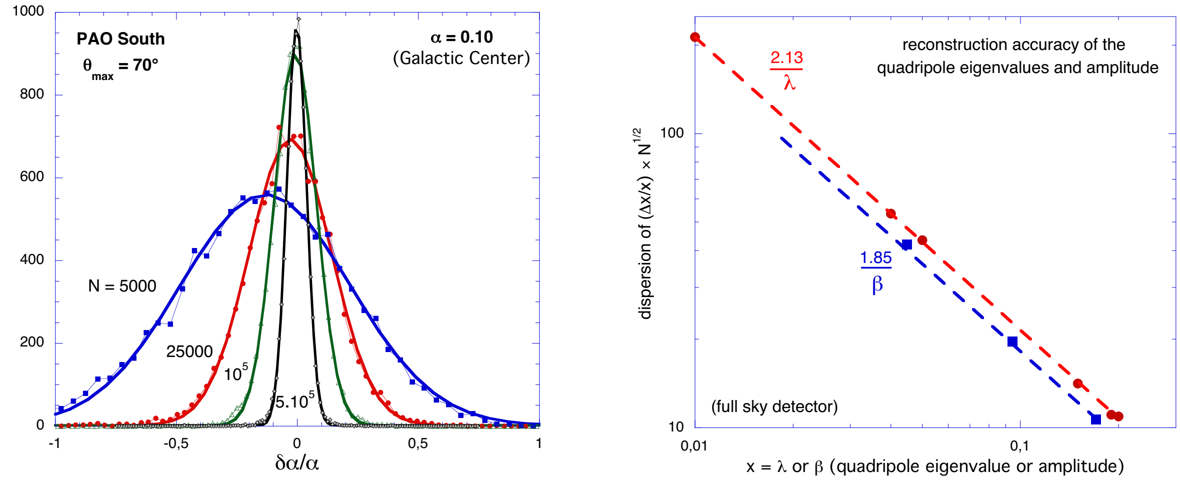

Figure 1. Left: histograms of the error on a pure dipole amplitude reconstruction, for various sizes of the data set. Right: Evolution of the accuracy with the anisotropy amplitudes (a uniform sky exposure is assumed).

N is the total number of CRs in the data set. This rewriting involves the exposure, $\mathcal{E}(\mathbf{u})$, in $\mathrm{m}^2\mathrm{s}$, achieved by the experiment under consideration in direction $\mathbf{u}$, since the actual probability to detect a CR from any given direction is proportional to $\Phi(\mathbf{u}) \times \mathcal{E}(\mathbf{u})$. Discrete versions of integrals (5) are thus (see [2]):

$$S_0 = \sum_k \frac{1}{\mathcal{E}(\mathbf{u}_k)} \quad , \quad \mathbf{S} = \sum_k \frac{\mathbf{u}_k}{\mathcal{E}(\mathbf{u}_k)} \quad \text{and} \quad S_{ij} = \sum_k \frac{(u_i)_k (u_j)_k - \frac{1}{3}\delta_{ij}}{\mathcal{E}(\mathbf{u}_k)} \tag{4}$$

where δ_{ij} is the Kronecker symbol. These nine discrete sums can be computed straightforwardly from the data, provided that the sky exposure is known (which simply derives from the detector's aperture). The derivation of the dipole and quadripole parameters then follows directly by identifying the sums with I_0, $\mathbf{I}$ and I_{ij}:

$$\alpha\mathbf{D} = 3\frac{\mathbf{S}}{S_0} \quad \text{and} \quad Q_{ij} = \frac{15}{2}\frac{S_{ij}}{S_0} \tag{5}$$

The Q_{ij} matrix can then be diagonalised numerically to obtain the quadripole eigenvalues and eigenvectors. As can be seen, when the CR observatory is able to observed the entire sky, the dipole and quadripole parameters are independent, which results from the orthogonality of spherical harmonics of different order. This is no longer true for partial sky coverage experiments, as the different modes are then mixed in a way that depends on the exposure function, $\mathcal{E}(\theta,\phi)$. It is also obvious that Eqs. (eq:SumsFullSky) cannot be implemented in direction where the exposure in null. On the other hand, $\mathcal{E}(\mathbf{u}_k)$ is never zero, by definition, since $\mathbf{u}_k$ is an actual CR arrival direction. The above procedure can thus be generalised by limiting the integration of the flux moments I_0, $\mathbf{I}$ and I_{ij} (to which the discrete versions S_0, $\mathbf{S}$ and S_{ij} should be identified), to the part of the sky that is actually observed (see [2] for details). Assuming that the part of the sky where the exposure is non-zero is contained between declination $\delta_{\min}$ and $\delta_{\max}$ (in equatorial coordinates), corresponding to spherical θ coordinates (as measured from the North Pole) between $\theta_{\min}$ and $\theta_{\max}$, one can write:

$$[\ I_0\ ;\ \mathbf{I}\ ;\ I_{ij}\ ] = \int_{\theta_{\min}}^{\theta_{\max}} d\theta\, \sin\theta \int_0^{2\pi} d\varphi\, [\ \Phi(\mathbf{u})\ ;\ \mathbf{u}\Phi(\mathbf{u})\ ;\ u_i u_j \Phi(\mathbf{u})\ ] \tag{6}$$

The dipole and quadripole parameters can still be obtained by identifying these integrals, computed analytically

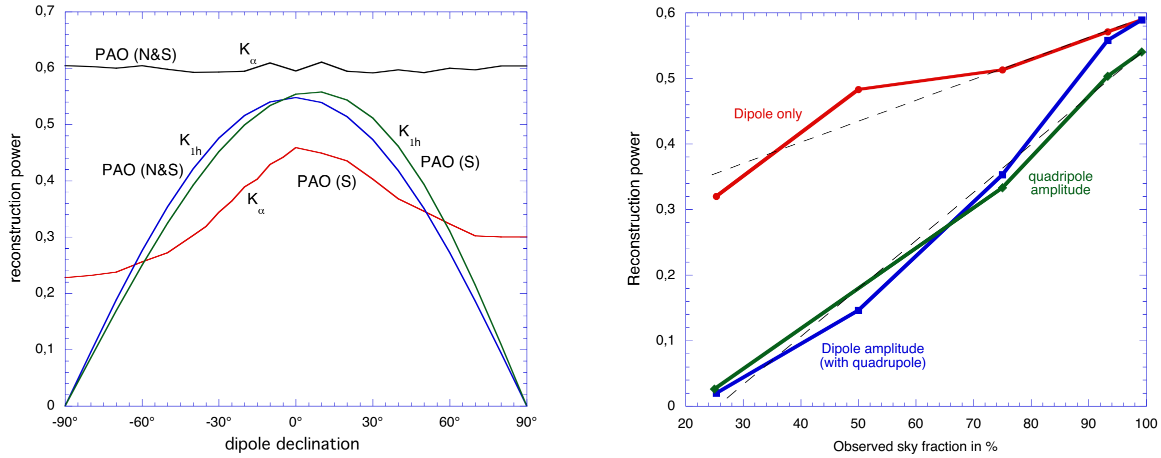

Figure 2. Left: dipole amplitude reconstruction power as a function of dipole declination, with either one or two PAO sites, compared with the power of the Rayleigh method (see text). Right: K_α and K_β evolution with the observed sky fraction.

replacing $\Phi(\mathbf{u})$ from Eq. (1), with the corresponding discrete sums over CR arrival directions (just as with full sky coverage). However, the inversion of the resulting linear system now introduces a mixing of the different modes, $l = 0, 1, 2$. For lack of space, we cannot give the results here (see ref. [2] for the case of a pure dipole reconstruction, and a forthcoming paper or the accompanying poster for the generalisation to a quadripole).

3. Reconstruction accuracy

To evaluate the accuracy of the method, we generated artificial data sets of various sizes, drawing CR arrival direction randomly according to a flux with known dipolar and quadripolar modulations (Eq. 1), and taking into account the relative exposure of the different parts of the sky (for any particular experiment of our choice). We then built the sums S_0, $\mathbf{S}$, and S_{ij} and reconstructed the anisotropy parameters blindly (cf. [1]), comparing them with the actual dipole and quadripole amplitudes and directions. We built histograms of the errors using 5000 different data samples for each set of anisotropy parameters. A systematic study thus allowed us to derive a simple general law giving the accuracy of the anisotropy reconstruction as a function of the dipole and quadripole amplitudes, α and β, and of the size of the data set, N.

For example, we studied the distribution of $\delta\alpha/\alpha = (\alpha_{\mathtt{true}} - \alpha_{\mathtt{rec}})/\alpha_{\mathtt{true}}$ as a function of N, for various values of the dipole amplitude and orientation. As can be seen in Figure. 1a, it is close to Gaussian even for relatively small data sets, which allows a simple characterization of the bias and uncertainty of the method. The mean and dispersion of $\delta\alpha/\alpha$ are found to converge to zero as N^{-1} and $N^{-1/2}$, respectively, as expected from statistics. Note that the bias is always smaller than the dispersion. In addition, it is found that for a given value of N the accuracy of the parameter reconstruction is inversely proportional to the anisotropy amplitude or eigenvalues, $x = \alpha$ or β or λ (see Figure. 1b), which allows us to write the significance of a given measurement as:

$$\Sigma \equiv \text{“number of sigmas”} = [\sigma(\delta x/x)]^{-1} = K_x \times x \times \sqrt{N}, \qquad (7)$$

where K_x is a pure number, which we call the *reconstruction power* (for the quantity x) of the CR observatory under consideration. It depends on its actual exposure function and fully characterizes its ability to measure CR anisotropies with the above-mentioned method. It should be noted, however, that in the case of a non-uniform sky coverage the reconstruction powers depend on the orientation of the dipole (or quadripole)[2]. This is shown in Figure. 2, where K_α was computed for the specific exposure functions achieved by the Pierre Auger Observatory (PAO), with either one site (South) or two sites (North and South). The evolution of K_α with the dipole declination is also compared with the equivalent reconstruction power for the first harmonic amplitude, $K_{1\mathrm{h}}$, as obtained with the standard Rayleigh method analysing the CR distribution in right ascension only.

Figure 2 also shows the evolution of the dipole and quadripole amplitude reconstruction powers as a function of the fraction of the sky that they can observed by the observatory (assuming a constant exposure in the declination interval $[-90°, \delta_{\mathrm{max}}]$, for illustrative purposes). As expected, the anisotropy reconstruction is always more efficient when this fraction is larger. The upper curve shows the reconstruction power K_α in the so-called "pure dipole" case, i.e. when there is no flux modulation of order more than 1, and when we know it in advance. The values of K_α are then relatively high, even with a limited sky coverage. However, when one does not assume that there is no quadripole and one tries to reconstruct it (even if it is actually zero), the limited sky coverage implies a mixing of the different modes, which results in a degradation of the dipole reconstruction accuracy. As can be seen in Figure. 2b, the reconstruction powers of the dipole and the quadripole are then similar ($K_\alpha \simeq K_\beta$), which is due to the Poissonian fluctuations of the sums S_{ij} (see above) entering in the global reconstruction of α, $\mathbf{D}$, and the Q_{ij} matrix.

The same study can be made for the reconstruction accuracy of the anisotropy directions (dipole and quadripole eigenvectors). The same general law as in Eq. (7) is obtained, with a dispersion of the reconstructed directions given by $\sigma_\theta = 1/(K_\theta\,\alpha\,\sqrt{N})$, where K_θ is the angular reconstruction power, in rad^{-1} (similar expressions hold for the quadripole, replacing α by β or λ. Figure 3 shows K_θ as a function of the observed sky fraction, in the case when both the dipole and quadripole are reconstructed (lower curve) and when anisotropies of order $l \geq 2$ are assumed negligible (upper curve). As an example, for a full-sky CR detector, $K_\theta \simeq 0.6\,\mathrm{rad}^{-1}$, so that a dipole of amplitude $\alpha = 10\%$ is reconstructed with a precision of $\simeq 9.5°$ with $N = 10^4$ events. With the same data set, the dipole amplitude would be measured at $\simeq 6\sigma$. With the same parameters and a detector covering only half of the sky, the precision on the dipole direction would be ~ 2 times lower if it is known (but how?) that no quadripole is present, and ~ 6 times lower otherwise. Similarly, the measurement of the dipole amplitude would be at the level of $\sim 4.5\sigma$ in the first case, and $\sim 1.6\sigma$ in the second case (see Figure. 2b).

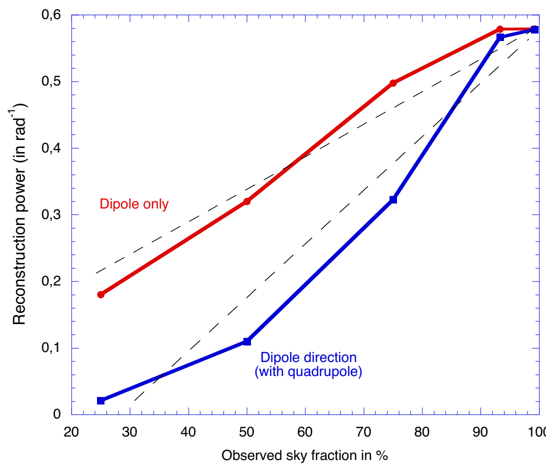

Figure 3. Same as Figure. 2b, but for the dipole angular reconstruction power, K_θ.

Please see ref. [2] for greater detail about the method and comparisons with other methods (see also [3]). A paper is in preparation for greater details on the quadripole reconstruction and comments about the coupling between anisotropy modes of different orders.

References

[1] Sommers, P., 2001, Astropart. Phys., 14, 271
[2] Aublin, J., & Parizot, E., 2005, A&A, in press
[3] Mollerach, S., & Roulet, E., 2005, (astro-ph/0504630)