ICRR,
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Uncertainty Relation and Weak Value

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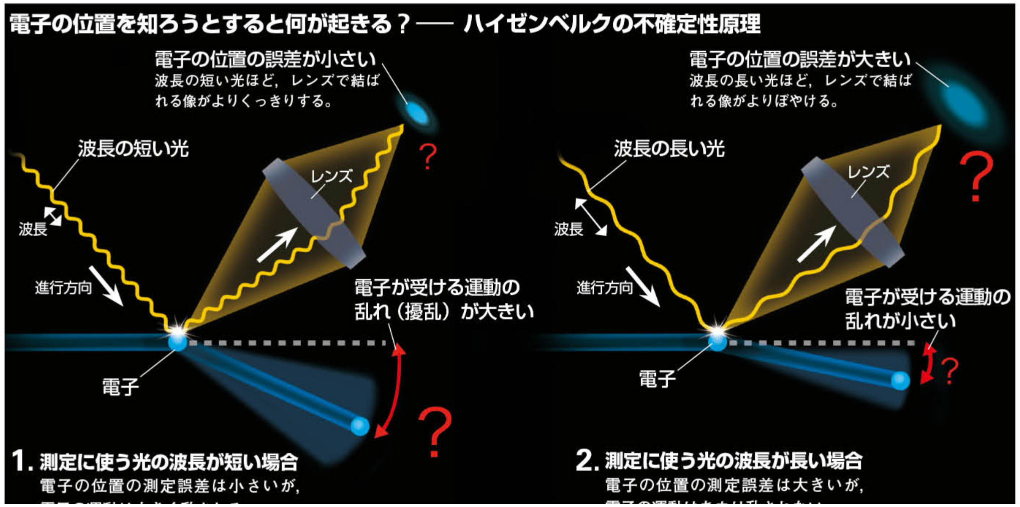
arXiv1104.1873
J.Phys.A44:415303,2011
with Minoru Koga

1. Uncertainty Principle

In the early 20th century Bohr, Heisenberg, Schroedinger and other people hotly discussed possible motion of electron in an atom, when it makes a transition between two quantum states.

Burkhardt Drude, one of Heisenberg's classmates even proposed a gamma-ray microscope to settle the issue.

W. Heisenberg, "DER TEIL UND DAS GANZE"(1969)



“Newton”
(2012)

$$\varepsilon(q)\,\eta(p) > \quad h/4\pi$$

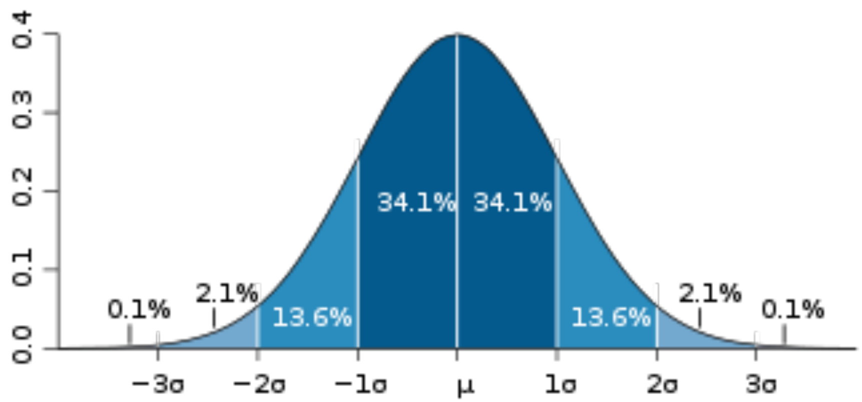
This motivated Heisenberg to examine the gedanken experiment for the error $\varepsilon(q)$ in the measurement of the position q of an electron and the disturbance $\eta(p)$ of its momentum p by the measurement of q. He concluded that we cannot define any “trajectory of an electron”.

W. Heisenberg: Z.Phys.43,172 (1927)

This inequality should not be confused with the Kennerd-Robertson inequality (1928),

$$\sigma(q)\, \sigma(p) \geq h/4\pi \quad (\#)$$

where $\sigma(q)$ is the standard deviation of the initial wavefunction before measurement.



Unfortunately many textbooks mixed up these two.
(Dirac mentioned only # in his textbook but carefully.)

Ozawa reformulated Heisenberg's uncertainty principle, on the basis of rigorous measurement theory of the Completely Positive (CP) map.

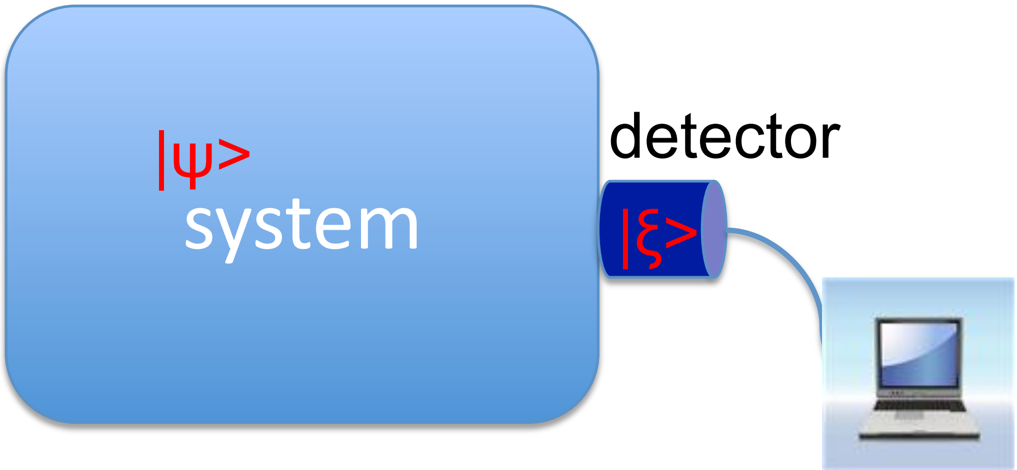
$$\varepsilon(A)\,\eta(B) + \varepsilon(A)\sigma(B) + \sigma(A)\,\eta(B) \geq |\langle\psi|[A,B]|\psi\rangle|/2$$

where $\varepsilon(A)$ is the error in the measurement of A, $\eta(B)$ is the disturbance of B by the measurement of A.
$\sigma(A)$ and $\sigma(B)$ are the quantum fluctuations of A and B in the state $|\psi\rangle$ i.e., the standard deviations.

M.Ozawa: Phys.Rev. A67,042105 (2003)

(sketch of proof)

The quantum measurement theory based on Complete Positive Map shows all physical measurement can be described by an unitary transformation U defined in the Hilbert space $H = H_{system} \otimes H_{detector}$ of the system and detector.



M. Ozawa, *Ann. Phys. (N.Y.)*, ***311, 350–416 (2004).***
See also Chuang and Nielsen textbook

CP map

A density operator ρ is called a positive operator, since the eigen values are non-negative because of the probabilistic interpretation. Any map which sends positive operators to positive operators is called a positive map. Obviously physical operation Λ to a state has to be a positive map.

It is natural to further demand that Λ is completely positive in the sense that

$$(\Lambda \otimes 1)(\rho \otimes \sigma) \geq 0 \quad \text{for all } \sigma.$$

The map Λ should be positive even if there are unaffected experimental apparatuses outside your own laboratory.

This seemingly obvious requirement is actually very strong and has been shown to be equivalent to

(1) Kraus representation

$$\rho \rightarrow \Lambda(\rho) = \sum_i A_i \rho A^*_i \ ,$$

with $\sum_i A^*_i A_i = 1$. The probability of the measurement outcome i is

$$p_i = Tr[\rho E_i],$$

$$E_i := A^*_i A_i$$ Positive Operator Valued Measure (POVM)

(2) Existence of the unitary operator in the measurement model of the system and detector such that $A_i = \langle \xi | U | i \rangle$.

Let A be an observable acting on H_{system} we are going to measure. Introduce a detector observable M on $H_{detector}$

We look at the difference of the time-evolved pointer variable M and the physical observable A before measurement,

$$E(A) := U^{+}(1 \otimes M)U - A \otimes 1.$$

Define the error $\varepsilon(A)$ in the measurement of A by M as

$$\varepsilon(A)^2 = \langle\psi| \otimes \langle\xi| E(A)^2 |\psi\rangle \otimes |\xi\rangle$$

The change of another observable B by the measurement of A is

$$D(B) := U^{+}(B \otimes 1)U - B \otimes 1$$

so that the disturbance $\eta(B)$ is defined as

$$\eta(B)^2 = \langle\psi| \otimes \langle\xi|\, D(B)^2 |\psi\rangle \otimes |\xi\rangle$$

$0=U^+ [B\otimes 1, 1\otimes M]\, U$

$=[U^+ (B\otimes 1)\, U, \; U^+ (1\otimes M)\, U$

$=[D(B)+B\otimes 1, E(A)+A\otimes 1]$

$=[D(B), E(A)]+[D(B), A\otimes 1]+[B\otimes 1, E(A)]+[B\otimes 1, A\otimes 1]$

Then

$[D(B), E(A)]+[D(B), A\otimes 1]+[B\otimes 1, E(A)]=[A\otimes 1, B\otimes 1]$

By the triangular inequality, we have

$$|| [D(B), E(A)] ||+||[D(B), A\otimes 1]||+|| [B\otimes 1, E(A)]|| \geq || [A\otimes 1, B\otimes 1] ||$$

where $||\#||=|\langle\psi|\otimes\langle\xi|\, \#|\psi\rangle\otimes|\xi\rangle|$

The Kennerd-Robertson inequality implies

$$\varepsilon(A)\,\eta(B) \geq |\langle\psi|\otimes\langle\xi|\,[D(B), E(A)]\,|\psi\rangle\otimes|\xi\rangle|/2$$

and so on, which we use three times to obtain Ozawa's inequality,

$$\varepsilon(A)\,\eta(B) + \varepsilon(A)\sigma(B) + \sigma(A)\,\eta(B) \geq |\langle\psi|[A,B]|\psi\rangle|/2$$

Recently, Ozawa's inequality is experimentally verified by Hasegawa's group in Wien in the neutron spin case.

J.Erhart et.al: Nature Phys. 8 185 (2012)

I would like to call your attention to the fact that the error and the disturbance are experimentally accessible if Ozawa's definitions are accepted.

However, there are some criticism on the definition of error and disturbance.

Here I will argue that the definition of error and disturbance actually leads to a deeper understanding of quantum mechanics that the "true value" of an observable is the weak value.

Preparatories

Look at the expression for the error

$$\varepsilon(A)^2 = \langle\psi|\otimes\langle\xi|(U^+(1\otimes M)U - A\otimes 1)^2|\psi\rangle\otimes|\xi\rangle$$

$$= \langle\psi|\otimes\langle\xi|(U^+(1\otimes M^2)\,U\,|\psi\rangle\otimes|\xi\rangle$$
$$- \langle\psi|\otimes\langle\xi|U^+(1\otimes M)UA\otimes 1\,|\psi\rangle\otimes|\xi\rangle - c.c.$$
$$+\langle\psi|A^2|\psi\rangle,$$

Let

$$U\,|\psi\rangle\otimes|\xi\rangle = \sum_m U_m|\psi\rangle\otimes|m\rangle,$$

where $M|m\rangle = m|m\rangle$

$$\langle\psi|\otimes\langle\xi|(U^{+}(1\otimes M^{2})\,U\,|\psi\rangle\otimes|\xi\rangle$$

$$=\Sigma_m\langle\psi|U^{+}{}_m m^2 U_m|\psi\rangle \qquad =\langle\psi|O_A{}^2|\psi\rangle$$

$$\langle\psi|\otimes\langle\xi|U^{+}(1\otimes M)UA\otimes 1\,|\psi\rangle\otimes|\xi\rangle$$

$$=\Sigma_m\langle\psi|U^{+}{}_m m U_m A|\psi\rangle \qquad =\langle\psi|O_A A|\psi\rangle$$

For the projection measurement $U_{m,}$ we write

$$O_A:=\Sigma_m m U_m$$

and use $O_A\otimes 1$ as a substitute of $1\otimes M$ of the pointer.

Then

$$\varepsilon(A)^2 = \langle\psi|(O_A - A)^2|\psi\rangle.$$

Similarly

$$\eta(B)^2 = \Sigma_m \langle\psi|([U_{m,} B]^2|\psi\rangle.$$

Hasegawa et al. measured the error and disturbance for $A=\sigma_x$ and $B=\sigma_{y.}$ O_A is a linear combination of the Pauli matrices.

We parameterize it as $O_A = \cos\varphi\ \sigma_x + \sin\varphi\ \sigma_y$

How can we measure $\varepsilon(A)$ and $\eta(B)$?

$$\varepsilon(A)^2 = \langle\psi|(O_A - A)^2|\psi\rangle$$
$$= \langle\psi|O_A^2 + A^2 - O_A A - A O_A|\psi\rangle$$
$$= 2 + \langle\psi|O_A|\psi\rangle + \langle A\psi|O_A|A\psi\rangle - \langle(1+A)\psi|O_A|(1+A)\psi\rangle$$

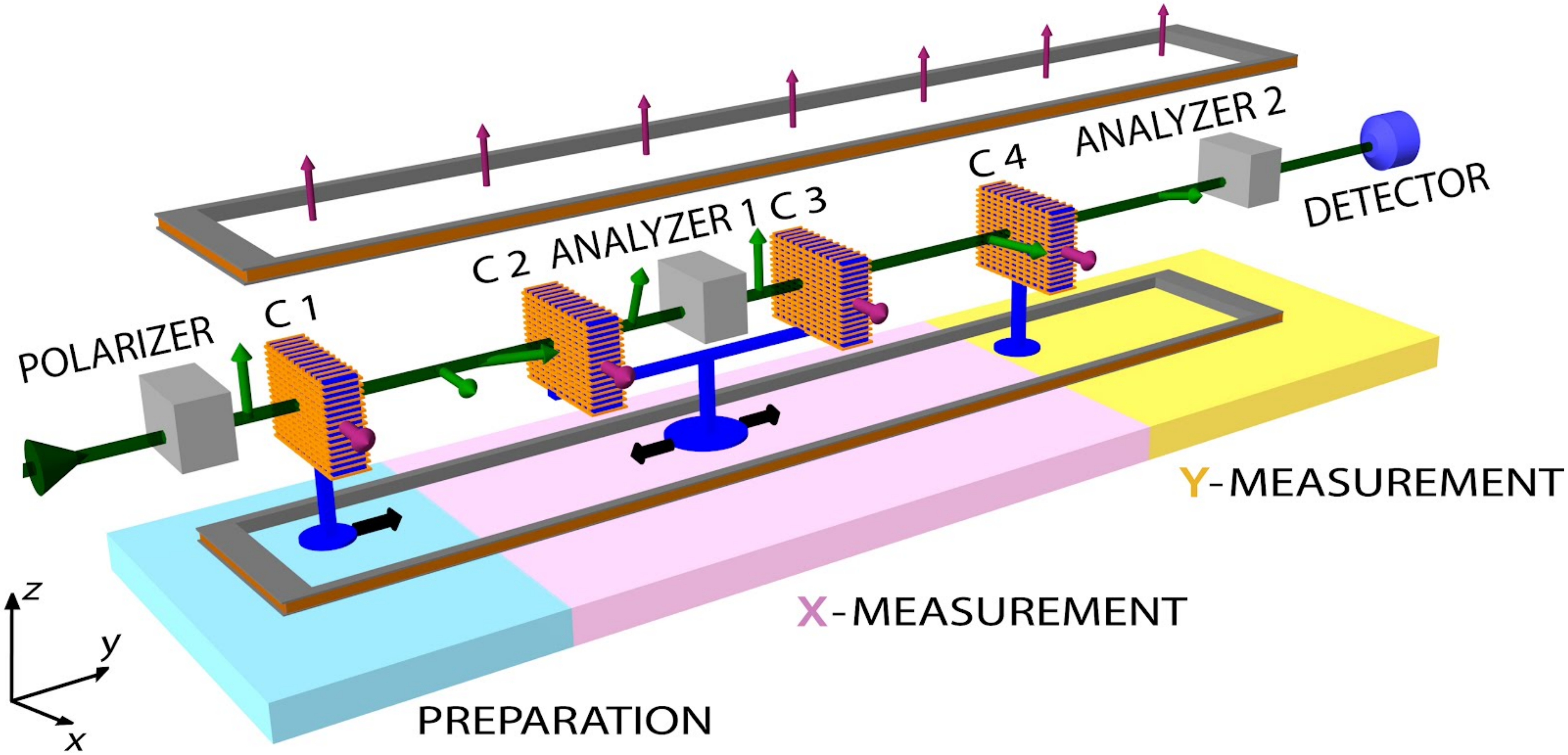
Each term can be evaluated by a sequential spin measurement. For example,

$$\langle\psi|O_A|\psi\rangle = [(I_{++} + I_{+-}) - (I_{-+} + I_{--})]/[I_{++} + I_{-+} + I_{+-} + I_{--}]$$

I_{-+} = # of neutrons when the 1st polarizer selects down-spin and the 2nd one up-spin.

$|A\psi\rangle = |\sigma_x\psi\rangle$ is the 180 degree rotated state of $|\psi\rangle$ and $|(1+A)\psi\rangle$ is the +1state along the x-axis.

All the terms can be evaluated by sequential spin measurements with appropriate rotation of the quantization axis.



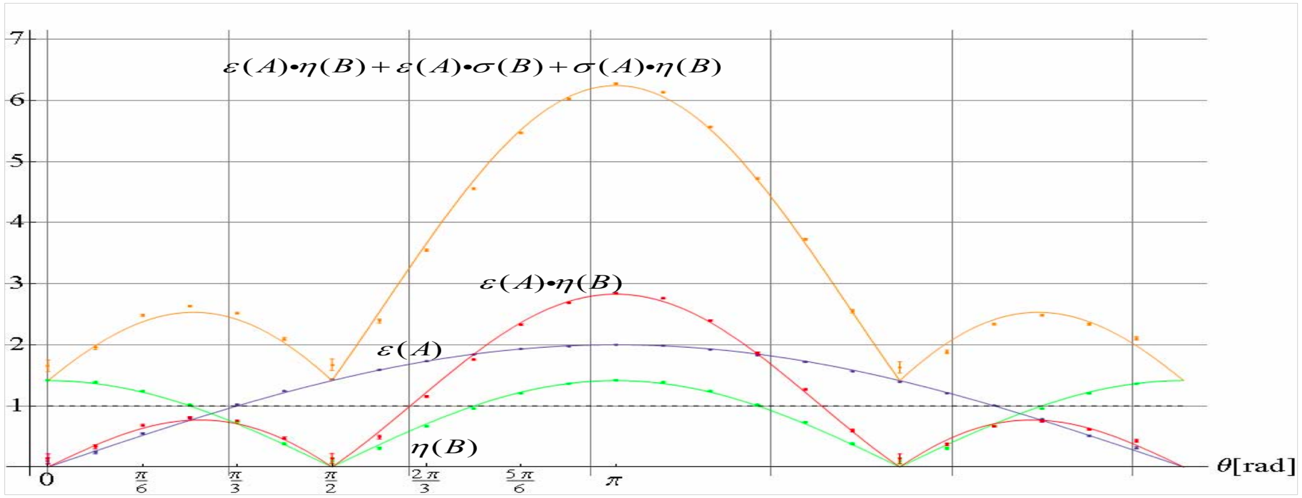
$$\varepsilon^2(A) = \langle (U(1 \otimes M)U^{+} - A \otimes 1)^2 \rangle$$

$$\eta^2(B) = \langle (U(B \otimes 1)U^{+} - B \otimes 1)^2 \rangle$$

where M is the meter operator to measure A and U is the unitary operator for the system and detector.
$\langle \ldots \rangle$ means the expectation value w.r.t. the initially prepared state $|\psi\rangle \otimes |\xi\rangle$ of the system and detector.

Hereafter we concentrate on $\varepsilon^2(A)$.

Results of the experiment, stand 13.11.209 By Yuji



Generally valid uncertainity relation of σ_x and σ_y :

$\varepsilon(\sigma_x)\bullet\eta(\sigma_y)+\varepsilon(\sigma_x)\bullet\sigma(\sigma_y)+\sigma(\sigma_x)\bullet\eta(\sigma_y)\geq|\langle S_z\rangle|$

In the experiment, the incident spin-state $|+z\rangle$ is measured by A (as a σ_x measurement), followed by the measurement of B (as a σ_y measurement):

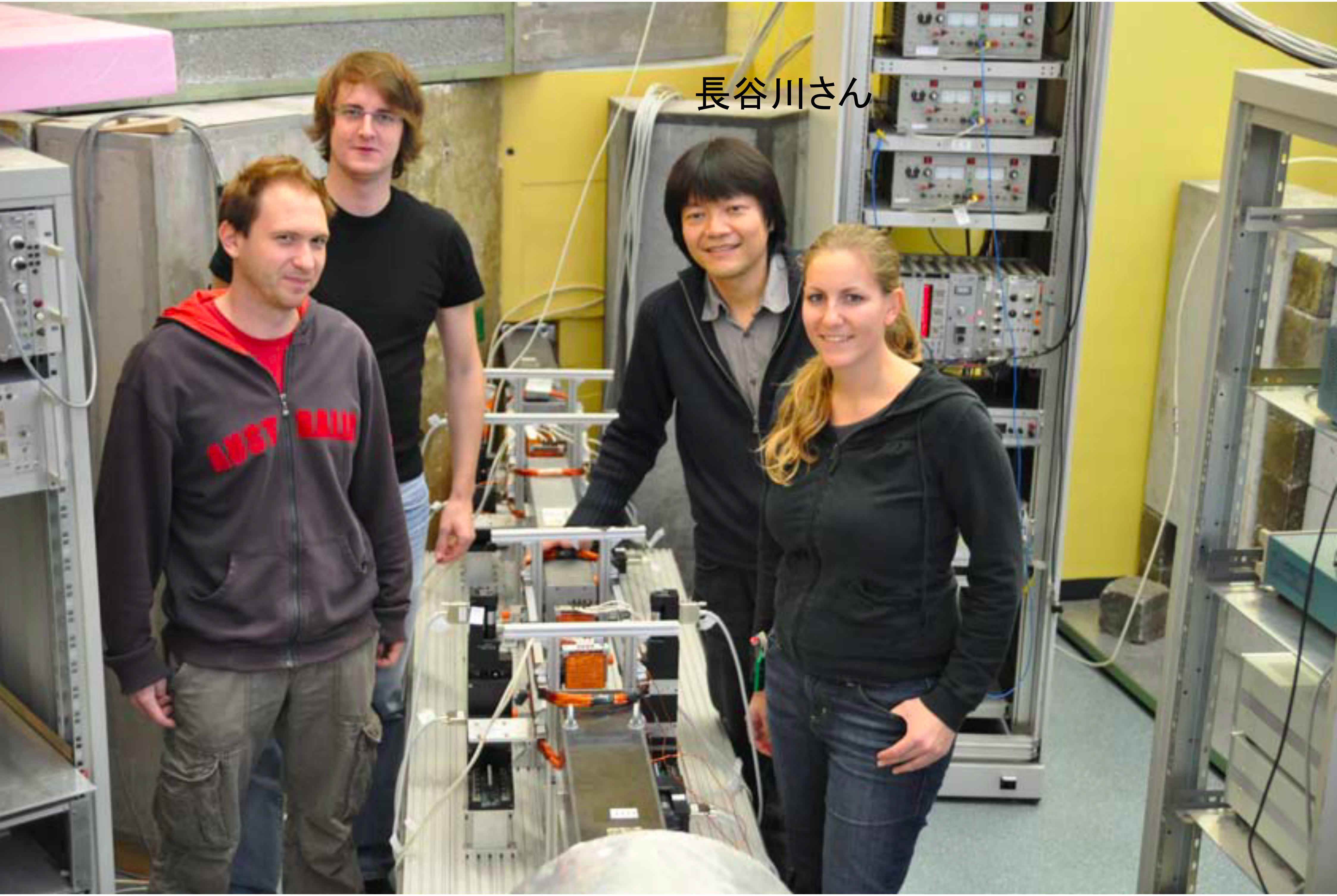
where $\begin{cases} A(\theta)=\cos\theta\bullet\sigma_x+\sin\theta\bullet\sigma_y \\ B=\sigma_y \end{cases}$

We measured $\varepsilon(A)$, $\eta(B)$, $\sigma(A)$ and $\sigma(B)$ and

$\varepsilon(A)$, $\eta(B)$, $\varepsilon(A)\bullet\eta(B)$, and $\varepsilon(A)\bullet\eta(B)+\varepsilon(A)\bullet\sigma(B)+\sigma(A)\bullet\eta(B)$ are plotted above.

Clearly seen is

$\varepsilon(A)\bullet\eta(B)+\varepsilon(A)\bullet\sigma(B)+\sigma(A)\bullet\eta(B)\geq\sqrt{2}(theoretical)>|\langle S_z\rangle|=1$


長谷川さん

Therefore, they concluded that Ozawa's version of uncertainty relation has been veryfied, while the original version is violated.

Since then there have been debates on the definition of error and disturbance.

e.g. ,Yu Watanabe (Ph.D thesis,Univ.Tokyo)

For the purpose of finding the fundamental limit of error/disturbance relation for all possible measurements Ozawa's definition seems most natural. Even more, it opens up a new concept of quantum mechanics ---weak value--.

Very recently Steinberg announced that his group also verified Ozawa’s inequality in an optical set-up by using measuring the weak value.

A. Steinberg (Vaexjo talk, June 2012)

The hint can be found in the paper by Lund and Wiseman New J. Phys. 12 093011 (2010) quoted in the paper by Ozawa and Hasegawa et al. Ozawa found the same formua much earlier

They gave an expression for the error/disturbance in terms of the weak values

$$\varepsilon^2(A)=\langle(U(1\otimes M)U^{+}-A\otimes 1)^2\rangle=\Sigma_{\delta a}(\delta a)^2 P_{wv}(a+\delta a)$$

$$\eta^2(B)=\langle(U(B\otimes 1)U^{+}-B\otimes 1)^2\rangle=\Sigma_{\delta b}(\delta b)^2 P_{wv}(b+\delta b)$$

Here

$P_{wv}(\delta a):=\Sigma_a\langle(U(1\otimes\Pi(a+\delta a))U^{+}(\Pi(a)\otimes 1)\rangle$

is the weak probability, where $\Pi(a)=|a\rangle\langle a|$ is the projection operator to the eigen state |a> of the operator A.

Note that in general

$\langle AB\rangle=\Sigma_\omega\langle A\Pi(\omega)B\rangle=\Sigma_\omega\langle A\Pi(\omega)B\rangle$

$=\Sigma_\omega\lambda^*_\omega(A)\,\lambda_\omega(B)P(\omega)$

can be expressed in terms of the weak values which can be measured with an arbitrary acuracy .

The error $\varepsilon(A)$ can also be rewritten as

$$\varepsilon^2(A)=\Sigma_{\omega,m} P(\omega,m)|A_m-\lambda_{\omega m}(A)|^2$$

(Hofmann, different from Lund Weisman's)

$$P(\omega,m)=|\langle\omega|K_m|\Psi\rangle|^2$$

is the probability to obtain the post-selected state $\langle\omega|K_m$ according to the standard Born's rule.
A_m is the measured value of A defined by

$$A_m|m\rangle=M|m\rangle.$$

$$\lambda_{\omega m}(A):=\langle\omega|K_m A|\Psi\rangle/\langle\omega|K_m|\Psi\rangle$$

where K_m is the Kraus operator: $K_m=\langle 0|U|m\rangle$

Note that the error can be interpreted as the Gaussian mean of the difference of the measured value A_m and the "true value" of A, $\lambda_{\omega m}(A)$.

One can instantly notice that

$$\lambda_{\omega m}(A) := \langle\omega|K_m A|\Psi\rangle / \langle\omega|K_m|\Psi\rangle$$

is the weak value proposed by Aharonov and his collaborators.

Y. Aharonov, D. Z. Albert, and L. Vaidman, Phys. Rev. Lett. 60, 1351 (1988).

There is a deep connection between the value of physical quantity before measurement and the weak value.

I will show this in a somewhat axiomatic way assuming

(1) Linearity: $\lambda(\alpha A+\beta B)=\alpha\lambda(A)+\beta\lambda(B)$ $\alpha,\beta \in R$

(2) Product rule when restricted to the Abelian subalgebra:

$\lambda(ST)=\lambda(S)\lambda(T)$ ∼classical theory

for all $S, T \in V_{max}$

(3) The prepared state is $|\Psi>$ nothing else;

$\lambda(|\Psi><\Psi|)=1$,

$\lambda(|\Psi^{\perp}><\Psi^{\perp}|)=0$, for all $|\Psi^{\perp}>$ s.t. $<\Psi^{\perp}|\Psi>=0$

(4)The expectation value Ex[A] and the variance Var[A] be independent of the choice of CONS $\Omega=\{<\omega|\}_{\omega}$,i.e., $V_{max} \in \mathcal{V}(N)$.

2. Formal theory of value of an observable

2.1 Context (finite dimension)

Let $\mathcal{V}(N)$ be a set of Abelian sub-algebras of all observables N .There may be many choices of the sub-algebra V_1, V_2, V_3 ….. $\in \mathcal{V}(N)$.

Choose $V_{max} \in \mathcal{V}(N)$. We call V_{max} as a context. The idea is that the mutually commutable set of observables {P,Q,R,….} define a set of simultaneous eigenvectors of P,Q,R,….{<ω|}, which corresponds to the resultant states after the projective measurements of P,Q,R,…. .

★ Inspired by Doering's lecture 2010 at Nagoya

As Bohr frequently emphasized, the description of quantum experiment has to be classical. The word "classical" may be paraphrased by the eigenvalues of mutually commuting observables {P,Q,R,….} in the maximal subalgebra V_{max}

The way of description (=context) of experiments is characterized by the choice of V_{max}.

We are going to define the value of an observable A in the state $|\psi>$ in the context V_{max}, i.e.,. .$\{<\omega|\}$.

Corresponding to the choice of the Abelian sub-algebra $V_1, V_2, V_3 \ldots \in \mathcal{V}(N)$, we have a collection of orthonormal basis $\{<\omega|\}_1$, $\{<\omega|\}_2$, $\{<\omega|\}_3$, ……

We can think of the collection of the values of an observable A in the state $|\psi>$ in the context V_1, V_2, V_3 …..i.e., $\{<\omega|\}_1$, $\{<\omega|\}_2$, $\{<\omega|\}_3$…

We fix a maximal Abelian subalgebra $V_{max} \in \mathcal{V}(N)$ for the moment of discussion and therefore the context $\Omega := \{<\omega|\}_\omega$. We shall find an expression for the value of an observable A-→ $\lambda(A) \in C$, complex number.

2.2 Main Theorem

We demand that the "value" $\lambda(A) \in C$ of an observable $A \in N$ satisfies the following properties:

(1) Linearity:

$$\lambda(\alpha A+\beta B)=\alpha\lambda(A)+\beta\lambda(B) \qquad \alpha,\beta \in R$$

(2) Product rule when restricted to the Abelian subalgebra:

$$\lambda(ST)=\lambda(S)\,\lambda(T) \qquad \sim \text{classical theory}$$

for all $S, T \in V_{max}$

(3) The prepared state is $|\Psi\rangle$ nothing else;

$$\lambda(|\Psi\rangle\langle\Psi|)=1,$$

$$\lambda(|\Psi^{\perp}\rangle\langle\Psi^{\perp}|)=0 \quad \text{, for all } |\Psi^{\perp}\rangle \text{ s.t. } \langle\Psi^{\perp}|\Psi\rangle=0$$

The above reqirements via Riez's theorem lead to

$$\lambda(A)=Tr[WA]/Tr[W] \quad \leftarrow\text{---}(1) \quad \lambda(\mathbf{1})=1 \leftarrow\text{---} (3)$$

with

$$W=a|\Psi\rangle\langle\omega|+b|\omega\rangle\langle\Psi| \leftarrow\text{---}(2)(3)$$

where $\langle\omega|$ is a simultaneous eigenvector of $V_{max.}$

The formal classical probability theory a la Kolmogorov presupposes the probability measure $P(\omega)$ and $\lambda_\omega(A)$ the value of a physical quantity A for an event ω.
The expectation value $Ex[A]$ and the variance $Var[A]$ are given by

$$Ex[A]=\Sigma_\omega P(\omega)\, \lambda_\omega(A) \quad ,\text{with } Ex[1]=1,$$

$$Var[A]=\Sigma_\omega P(\omega)|\, \lambda_\omega(A)-Ex[A]|^2$$

We adopt these expressions also in quantum mechanics. Note that $P(\omega)$ is independent of A.

Correspondence with the classical probability theory.

$\omega \in \Omega$: event ($\leftrightarrow \langle\omega| \in V_{max} \in \mathcal{V}(N)$)
$dP(\omega)$: probability measure (independent of A) ($\leftrightarrow P(\langle\omega|)$)
$h_A(\omega)$: a random variable (real) ($\leftrightarrow \lambda_\omega (A)$: complex)

Expectation value and variance:

$$Ex(A) := \int dP(\omega)\, h_A(\omega)$$
$$Var(A) := \int dP(\omega)\, |h_A(\omega) - Ex(A)|^2$$

(4) We demand the expectation value Ex[A] and the variance Var[A] be independent of the choice of CONS $\Omega=\{\langle\omega|\}_{\omega}$, i.e., $V_{max} \in \mathcal{V}(N)$.

According to the central limit theorem, the distribution of values of observable A approaches the normal (Gaussian) distribution characterized by its mean Ex[A] and the variance Var[A].

The requirement (4) demands that the distribution should be independent of how we measure A but depends only on the prepared state $|\Psi\rangle$.

The above requirement uniquely determines both

$W=|\Psi\rangle\langle\omega|$

and therefore the "value" coincides with Aharonov's weak value

$\lambda_{\omega}(A)=Tr[WA]/Tr[W] = \langle\omega|A|\Psi\rangle/\langle\omega|\Psi\rangle$, (i.e., b=0)

Aharonov, Albert and Vaidman, PRL 60,1351

and the measure,

$P(\omega)=|\langle\omega|\Psi\rangle|^2$

and therefore we have "derived" the Born formula for the expectation value and the variance

$Ex[A]=\langle\Psi|A|\Psi\rangle$

$Var[A]=\langle\Psi|(A-Ex[A])^2|\Psi\rangle$.

Outline of proof

Introducing the lagrange multiplier μ to ensure the completeness relation, we demand the variation of the lagrangian $L[\langle x|,\mu]$ w.r.t. $\langle x|$ and μ vanish for all Observable A

$$L[\langle x|,\mu]=Ex[A]-\mu(\Sigma_x\langle\Psi|x\rangle\langle x|A|\Psi\rangle-\langle\Psi|A|\Psi\rangle)$$

$$=\Sigma_x P(x)\,\lambda_x(A)-\mu(\Sigma_x\langle\Psi|x\rangle\langle x|A|\Psi\rangle-\langle\Psi|A|\Psi\rangle)$$

where $\lambda_x(A)=Tr[WA]/Tr[W]$

$$W=a|\Psi\rangle\langle x|+b|x\rangle\langle\Psi|$$

Idea of the proof: if $P(\omega)=|\langle\Psi|\omega\rangle|^2$

$$Ex[A]=\Sigma_\omega P(\omega)\ \lambda_\omega(A) = \Sigma_\omega |\langle\Psi|\omega\rangle|^2 [\langle\omega|A|\Psi\rangle/\langle\omega|\Psi\rangle]$$
$$= \Sigma_\omega \langle\Psi|\omega\rangle\langle\omega|A|\Psi\rangle$$
$$=\langle\Psi|A|\Psi\rangle$$

$$Var[A]=\Sigma_\omega P(\omega)\ |\lambda_\omega(A)|^2 = \Sigma_\omega |\langle\Psi|\omega\rangle|^2 |\langle\omega|A|\Psi\rangle/\langle\omega|\Psi\rangle)\ |^2$$
$$= \Sigma_\omega \langle\Psi|A|\omega\rangle\langle\omega|A|\Psi\rangle$$
$$=\langle\Psi|A^2|\Psi\rangle$$

do not depend on $\{\langle\omega|\}$ i.e., the choice of $V_{max} \in \mathcal{V}(N)$.

Note that $P(\omega)=|\langle\Psi|\omega\rangle|^p$ would not work for $p\neq 2$!

The key is the completeness relation $\Sigma_\omega |\omega\rangle\langle\omega| = 1$.

c.f. **J.Phys. A;43 025304 (2010) with Shikano**

The weak value $\lambda_\omega(A)$ is a fundamental quantity in quantum mechanics which is experimentally accessible only slightly disturbing the prepared state e.g., by the weak measurements with an arbitrary accuracy in principle. For me, at least, it is almost a "physical reality" in the sense of EPR.

The probability measure $P(\omega)=|\langle\omega|\Psi\rangle|^2$ is not an axiom any more but a consequence of quantum mechanics and the probability theory.

$\lambda_\omega(A)$ is interpreted as a value of A in the context of the pre-selected state $|\Psi\rangle$ and the post-selected states $\{\langle\omega|\}$ of the intended projective measurements of a maximal set of commuting observables $V_{max} \in \mathcal{V}(N)$.

Since an event ω is identified with a consequence $\langle\omega|$ of measurement of V_{max} , $P(\omega)=|\langle\omega|\Psi\rangle|^2$ is interpreted as its relative frequency.

From the formula:

$Ex[A]=\Sigma_\omega P(\omega)\ \lambda_\omega\ (A)$

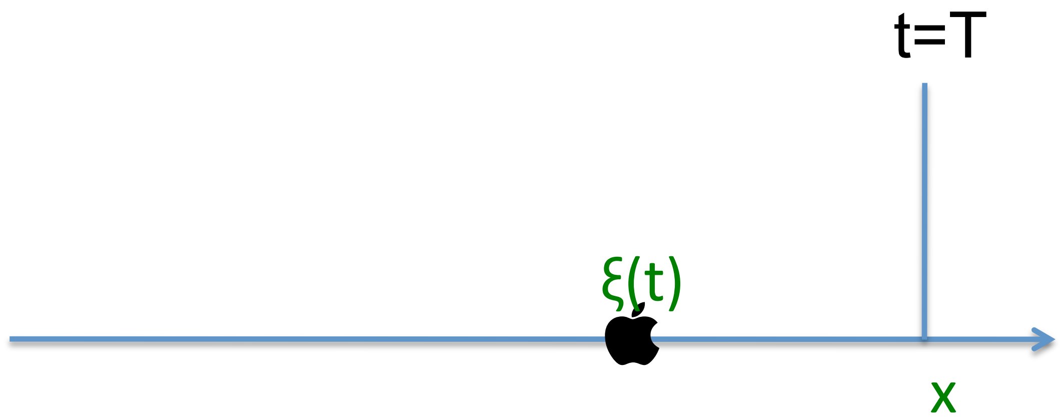
λ_ω $(|a\rangle\langle a|)$ can be interpreted as a conditional probability to reach $\langle\omega|$ via $|a\rangle\langle a|$ with the initially prepared state $|\Psi\rangle$. In general λ_ω may be complex.

J.Phys. A;43 025304 (2010) with Shikano

Going back to the original motivation of the value of an observable before measurement（“真値”）
we just show an example:

$$\xi(t) := \langle x|X(t)|\Psi\rangle / \langle x|\Psi\rangle,$$

where $X(t)$, $0 \leq t \leq T$ is the position operator of a particle. $\langle x|$ is the eigen state of $X(T)$ with the eigen value x.



We can ask the following counter-factual question.

We are in a certain initial state and
know the value of X as x by measuring X of
at t=T.
What the value of X(t) would be before T ?

We can answer in an experimentally verifiable way.

A remark:

☆ The weak value is almost a wave function itself
For the case that the momentum is conserved,

$\eta(t) := \langle x|P(t)|\Psi\rangle / \langle x|\Psi\rangle = i\Psi'(x)/\Psi(x)$

See the experiment which shows a trajectory of photon
by weak measurements of its transverse momentum

“Direct measurement of the quantum wavefunction”
by Lundeen et al. (Recent Issue of Nature),

Reducing the strength of the measurement corresponds to reducing α, which makes it impossible to discriminate with certainty whether any

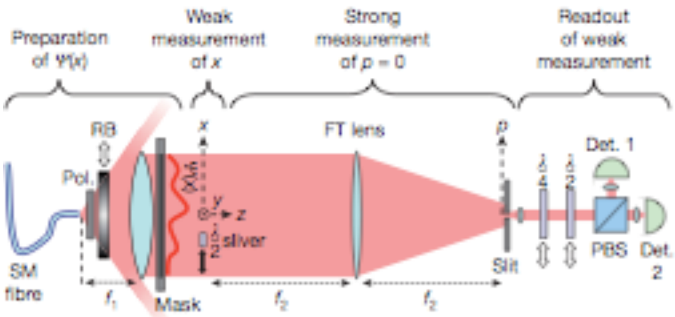
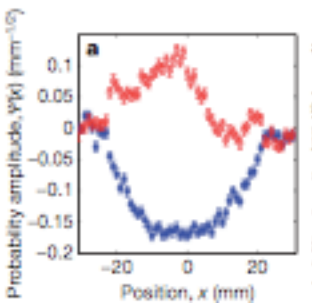


Figure 1 | Direct measurement of the photon transverse wavefunction. To begin with photons having identical wavefunctions, we transmit them through an optical fibre (Nufern PM780-HP) that allows only a single mode (SM) to pass. This mode is approximately Gaussian, with a nominal $1/e^2$ diameter of $5.3 \pm 1.0\,\mu m$. The photons emerge from the fibre and pass through a micro-wire polarizer (Pol.; Edmund Optic NT47-602) to be collimated by an achromatic lens (f_1 = 30 cm, diameter 5 cm, Thorlabs AC508-300-B), one focal length (f_1) away from the fibre. The lens was masked off with a rectangular aperture of dimension $x \times y$ = 43 mm × 11 mm. Thus our nominal initial wavefunction was a truncated Gaussian with a $1/e^2$ diameter of 56.4 mm and a

normalize the σ_x and σ_y measu $\int |\Psi(x)|^2 dx = 1$, which elimin sin $\alpha/\Phi(0)$.

To confirm our direct measure different wavefunctions. Using ou by measuring the initial trunca described in Fig. 1. Switching to modify the magnitude, and then with an apodized filter and glass test Ψ (Fig. 3). We conduct m wavefunction phase by introduci phase curvatures (Fig. 4). For all t



Lundeen et al.

There have been a hot debate in Japan over the “true value” of an observable before the post-selection.
Some people are still reluctant to accept the assertion that such a thing exists and actually coincides with the weak value.
One of the reason may be that the weak value depends on the choice of the post-selected state by an observer.
This does not fit the “objectivity” of the “truth”.

I would say that the weak value is the subjective value of objective physical quantity. In contrast the expectation value is an objective value.
Note that we cannot assign objective value for all the objective physical quantities as Kochen and Specker proved.

Kochen-Specker theorem ('67) (c.f. Zeilinger's recent paper)

We cannot assign a value of all physical quantities independently of how we measure it for dim(H)≥3.

Example by Mermin (4 dim)

$\sigma_x \otimes 1$	$1 \otimes \sigma_x$	$\sigma_x \otimes \sigma_x$	1
$1 \otimes \sigma_z$	$\sigma_z \otimes 1$	$\sigma_z \otimes \sigma_z$	1
$\sigma_x \otimes \sigma_z$	$\sigma_z \otimes \sigma_x$	$\sigma_y \otimes \sigma_y$	1
1	1	-1	

cannot assign the eigenvalues ±1 consistently

Eigenvalues are only non-contextual values

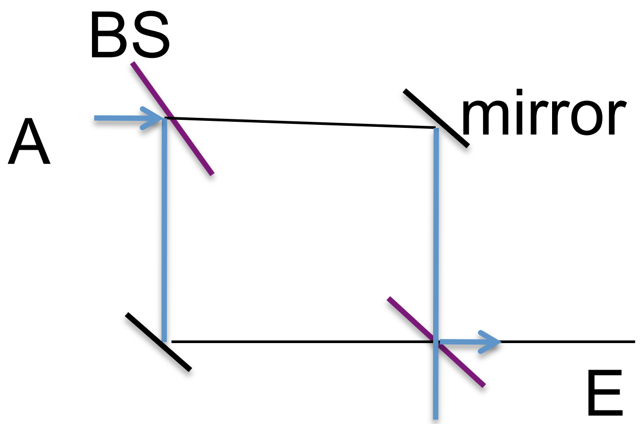
Mermin '90

The utmost important thing is that the “true value” of an observable before measurement can be experimentally verified by weak measurements.

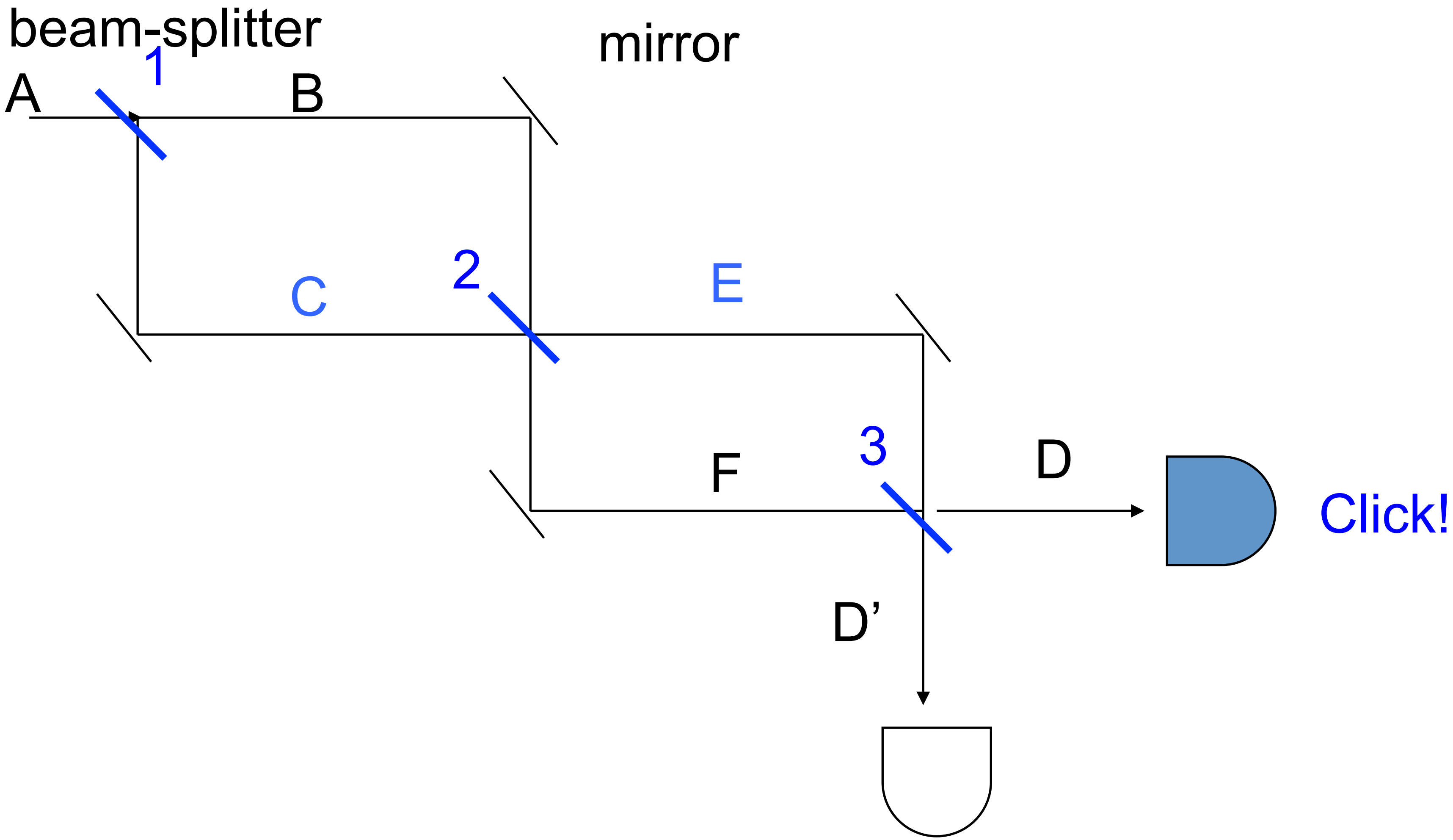
The value of observable certainly exists before measurement. The measurement only discovers it.

Mach-Zehnder interferometer

A photon injected from A goes out of E only by interference



Double MZ interferometer



Mitchison, Jozsa and Popescu   arXiv 0706.1508

We can see

A is pre-selected → photon would pass E
(interference)

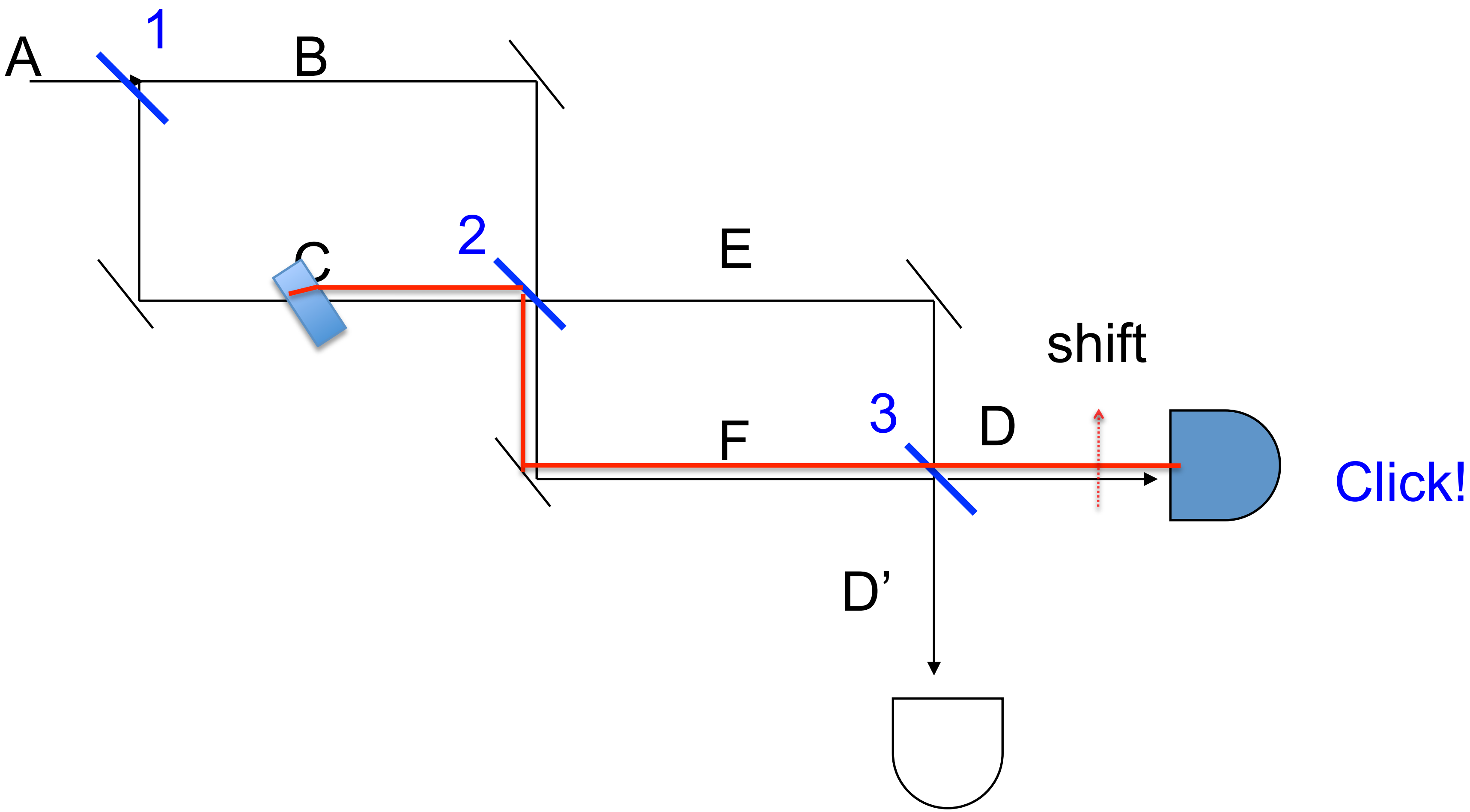
photon should have passed ← D is post-selected
(which way)

We can know which-way, while interference effect is manifested! (Aharanov)

Verfication by a weak mesaurement

Insert a tilted slide glass in the path C
The optical axis will shift.

(similar for E)



4. Summary of the 1st part

I reviewed the uncertainty relation by Ozawa and its recent verification by Wien group led by Hasegawa on the emphasis,

(1) The mathematical background (CP map)
(2) how the original mathematical expression have been made operational for the actual experiment.

I think both are very innovative! Furthermore this leads to an important concept of the weak value.

Summary of the 2nd part

Combining quantum mechanics and the formal probability theory we have shown that the context dependent (subjective) value of (objective)observable A is the weak value

$$\lambda_\omega (A) := \langle\omega|A|\Psi\rangle / \langle\omega|\Psi\rangle$$

and the probability measure is given by Born's rule:

$$P(\omega) = |\langle\omega|\Psi\rangle|^2,$$

where $|\Psi\rangle$ is the initial state and $\langle\omega|$ is the post selected state.

The value of observable appears only after measurement not before in the Copenhagen interpretation.

However,

Carl Friedrich von Weizsäcker denied that the Copenhagen interpretation asserted: "What cannot be observed does not exist". He suggested instead that the Copenhagen interpretation follows the principle: "What is observed certainly exists; about what is not observed we are still free to make suitable assumptions.

Thank you for your attention!